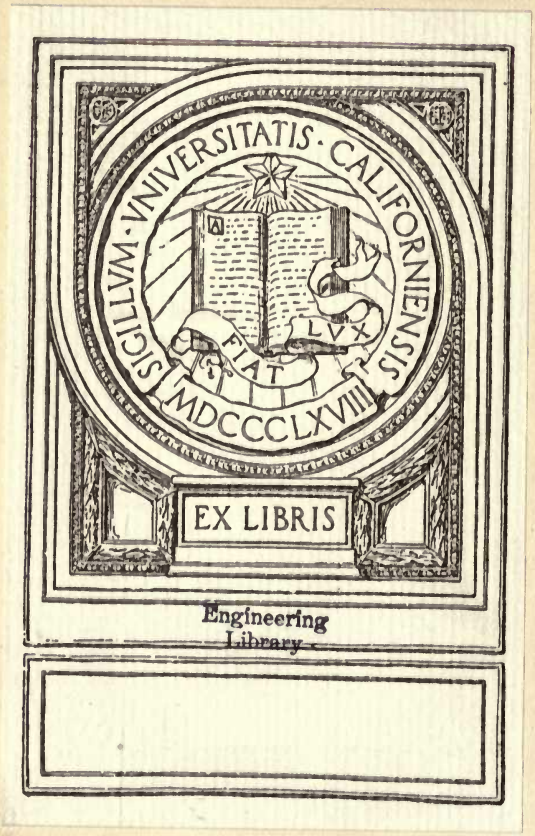

MASONRY DAM DESIGN

INCLUDING

HIGH MASONRY DAMS

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PREFACE TO SECOND EDITION

In this revision the authors have attempted to amplify some of the features which, in the earlier edition, were little more than touched upon. Chapters on the Overfall and Arched types of masonry dams have been added, together with cross-sections of a selected series of masonry dams chronologically arranged. The last are for the purpose of comparison and showing the development of the masonry dam from the time of the massive Spanish type to the present.

The design of low and medium-sized as well as that of high-masonry dams, may be prosecuted according to the theory and methods in this work, as general expressions have been written wherever possible. It was considered best to indicate this in the title while the original page captions remain the same.

The authors are indebted to Miss Bessie N. MacDonald, A.B., for assistance in verifying the more difficult mathematical derivations for the Arched Dam. Acknowledgment is also made to Mr. Alfred D. Flinn, Member American Society of Civil Engineers, who had kindly furnished one of the authors with a set of cross-sections of dams, a number of which are shown in the series in Appendix III.

C. E. M.
O. L. B.

New York City,
February, 1916.

PREFACE TO FIRST EDITION

It is the practice at Columbia University to require of the third-year students in the Department of Civil Engineering, the execution of the design of a masonry dam, and to aid them in this problem they have heretofore been furnished with "Notes on the Theory and Design of High Masonry Dams," prepared some years ago by Prof. Burr of the Department, and having for their basis the method as set forth by Mr. Edward Wegmann.

This procedure with which Wegmann is credited, and which was developed through the investigations undertaken in connection with the Aqueduct Commission of the city of New York, for the purpose of determining a correct cross-section for the Quaker Bridge dam, resulted in the first direct method for calculating the cross-section of such structures and is essentially a development of the Rankine theory.

The studies appeared first in the report made by Mr. A. Fteley to the chief engineer of the Aqueduct Commission of the city of New York, dated July 25, 1887, and later in Mr. Wegmann's treatise on "The Design and Construction of Dams."

Neither in the report nor in the treatise however, have the effects of uplift, due to water permeating the

mass of masonry, and of ice thrust, acting at the surface of the water in the reservoir, been considered, and in consequence of this, objection might be legitimately raised that the series of equations determining the cross-section fail to account for these factors. Some difference of opinion may exist as to the relative importance of these considerations, but when a structure of great responsibility is projected, conservatism in design is essential.

The following presentation which aims to supply these omissions, has been prepared primarily that there may be had in convenient form a text, containing the general treatment and such consideration of these factors as more recent practice requires, together with a brief statement regarding the late investigations undertaken for the purpose of determining more accurately the variation of stress in masonry dams.

The formulæ relating to uplift, ice thrust, etc., were deduced by one of the authors and have been used in part in connection with the design of the large dams for the new water supply for the city of New York.

The computations for the design of a high masonry dam are appended to facilitate the ready comprehension and application of the formulæ.

It is hoped that the presentation may appeal to the practicing engineer as well as the student, and that there may be found therein enough to compensate him for the labor involved in its perusal.

C. E. M.
O. L. B.

COLUMBIA UNIVERSITY, 1910.

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INTRODUCTION

THE method of analysis by which an economical cross-section of a gravity type high masonry dam may be most directly calculated, and the one which is most generally adopted in engineering practice, was first devised by Mr. Edward Wegmann through studies made for the Aqueduct Commissioners of New York City, in connection with the design of the New Croton Dam, and it is that method which will be employed here, though it will receive some modification in certain particulars and be elaborated in certain others.

In determining the cross-section by the series of equations developed in that analysis, no account is taken of uplift due to water penetrating the foundation or the mass of masonry above, nor of the ice thrust acting horizontally against the up-stream face of the dam, at the surface of the water in the reservoir, though reference is made to it. Present practice requires, however, that these two factors be recognized where a structure of great responsibility is proposed, and in this respect at least will the analysis be amplified.

MASONRY DAM DESIGN

Including High Masonry Dams

CHAPTER I

UPWARD PRESSURE AND ICE THRUST

Part I—Upward Pressure

Although it had been appreciated for a number of years that a complete analysis of a high masonry dam required the consideration of uplift and ice thrust, until comparatively recently no structure of this type had been designed which allowed for these two factors in the computations.

In fact, it may be said that prior to the year 1853 masonry dams were built without a rational consideration of *any* of the forces acting in or upon them, for it was not until then that de Sazilly first indicated the principles upon which dam design is based, by providing for a sufficient safety factor against sliding and overturning and by assigning a maximum limit of pressure against the crushing of the material.

Some time later Rankine added to the theory by prescribing the well-known requirement that the line of resultant pressure for reservoir, full or empty, should lie within the middle third of the structure, to preclude the possibility of tension in any joint, and suggested that the

limit of pressure should be made less for the down-stream edge than for the up-stream.

In 1884, when the Aqueduct Commission of New York City came to design the New Croton Dam, a structure between 275 and 300 feet high, then the highest in the world, and exceeding the next highest by about 100 feet, it was found necessary to modify some of the older conclusions with respect to dam design, in order to make the theory applicable to their particular problem.

Thus, where heretofore the prescribed limit of crushing strength of masonry had been assumed to be between 6 and 10 tons per square foot, they increased it to 16 tons, as it had been demonstrated that such pressures actually existed in dams still doing duty, and since, with the lower values, the computations would have given a horizontal face at a joint 300 feet below the top. Both upward pressure and ice thrust were considered, but both in turn were disregarded. The former because it was felt that the condition of the masonry and of the foundation was such that the entering of water would be a remote possibility, and the latter because it was believed that the mass of the masonry was sufficiently great to care for any additional forces due to the ice thrust.

It remained, therefore, for the engineers of the Wachusett dam in Massachusetts to be the first in the United States actually to incorporate uplift and ice thrust in the design of a high masonry dam. They may have been led to this precaution by the fact that the structure was located only one-half mile above a town of some 13,000 inhabitants, where a failure would result in enormous loss of life, and where it was, in consequence, necessary

to be particularly careful, but at any rate they were the first to adopt these two considerations in the design of a high masonry dam.

Their assumption for uplift was two-thirds of the static head at the up-stream edge, diminishing as a straight line to zero at the down-stream edge, and for ice thrust, 47,000 pounds per linear foot of dam, or equivalent to the crushing strength of ice one foot thick.

To-day, in the light of experience, no structure of this character would be built without careful consideration of both these elements, and it is doubtful if, under any circumstances, they would be eliminated entirely, though they might not receive the same weight they were given in the computations for the Wachusett Dam.

That engineers are not fully agreed on the matter of uplift and ice thrust and that a considerable diversity of opinion exists in the profession with respect to them, may perhaps be partially explained by the fact that the former does not lend itself to an exact treatment, while, with regard to the latter, there are no exact data as to the expansive force of ice acting at the surface of a reservoir. Furthermore, there are many high masonry dams now standing which were designed with no consideration being given to these two factors, and this would seem to refute the argument that they are necessary considerations for safety.

It is recognized, however, that the influence of upward pressure and ice thrust on the stability of masonry dams, together with the actual internal distribution of stress in very large masses of masonry are probably the most indefinite factors in the design of such structures.

Uplift.—It may not be out of place to explain here at some length what " uplift " means and how it may become active.

In a masonry mass, especially a large concrete mass, cracks can be formed by temperature changes, due to the setting of the concrete in the first place, and to subsequent daily and seasonal exterior temperature variations.

Contraction joints, provided to meet the effect of such temperature changes in the body of the masonry, are often built in large dams.

Discontinuity of the mass of a large masonry structure like a dam, owing to interruption and resumption of construction work from day to day, is also evidenced by joints, mostly horizontal, perhaps, but, in spite of the utmost attempts to preserve continuity, often unavoidable.

Temperature cracks, contraction and construction joints, then, all tend to affect permeability to a greater or less degree, admitting water to the body of a dam according to the pressure exerted by that water.

Besides, as it has been observed that water under sufficient head has passed through 30 feet thickness of good concrete and that under enormous pressures water has been made to ooze through cast-steel cylinders, it may be appreciated from the above considerations that water from a reservoir may enter the masonry mass of the dam. In fact, it has been frequently found in high masonry dams that, following construction and upon filling the reservoir, small issuing streams or leaks have appeared on the down-stream face. These leaks have been observed at

the base, down-stream, as well as higher up, so it may be safely assumed that water from the reservoir may penetrate the natural foundations as well as the masonry above, especially if the former consist of a porous or stratified formation.

Therefore, in a structure of considerable height, which retains a body of water behind it, there may be exerted a powerful vertical force acting upward under the dam or on some joint of the masonry above. This force is commonly termed "uplift," and will, of course, depend, for its amount, upon the hydrostatic head. Furthermore, this force, due to the water pressure, tends to counterbalance the downward, vertical component of all forces acting in or upon the structure.

Obviously, were this "uplift" to become sufficiently great it might actually float the structure off its foundation or off any joint, whereupon the horizontal water thrust back of the dam would complete the destruction by sliding it down-stream.

Upward pressure, therefore, should receive consideration, both from the standpoint of its effect in the foundation of the dam, and also in any of the joints above.

Naturally, it is much more difficult to ascertain the condition of this with respect to the foundation, as the latter's physical characteristics are never revealed until actual work has begun on the structure and the site is uncovered. For this reason, it should be made imperative to examine by exploration, drill holes, etc., as completely as possible, the nature of the foundation, so that its true state may be at least approximately known, and so that, also, proper provision for upward pressure may be made.

An examination of the outcropping rock at a dam site will never be sufficient to determine the nature of the foundation below, as the latter may not conform with the exposed surface. Core borings should be made at frequent intervals. They should be driven well into the bed rock to develop the character, and where there is limestone with a likelihood of cavities, particular care should be exercised. Upon such cavities being uncovered, they should be filled with grout and concrete, so as to preclude the entrance of water. Very porous sandstone or the existence of seams and strata may give rise to very dangerous conditions. Thus, an examination by borings at the Austin, Pa., dam site would have indicated the porosity of the rock, and might have been the means of preventing the disaster which followed. All foundations should be tested for tightness by applying air or water pressure to the drill holes.

Even in the best foundation, however, it may be said that there is no absolutely water-tight condition.

All water getting into the dam should be collected in a chamber or tunnel, carried outside and measured for quantity. Thereby a measure of the water-tightness of the dam may be ascertained.

Treatment of Uplift. — There is not the clearest conception among engineers as to how to allow for this upward pressure, but in some of the more recent discussions,* it has been suggested that perhaps three general conditions may be recognized.

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* "Provision for Uplift and Ice Pressure in Designing Masonry Dams." By C. L. Harrison, Trans. Am. Soc. C.E., Vol. LXXV, p. 142.

1. The case where no upward pressure could exist because the foundation rock, and the joints of the dam, were so tight that no water could possibly enter. Evidently for such a condition provision for upward pressure in the design of the dam is unnecessary.

2. Second, the case at the other extreme, where the rock is of such a nature that water may freely enter the foundation, and as freely leave it from the lower edge of the dam. Here it is quite evident that the water would enter with the full hydrostatic pressure acting at that point or elevation, and if the water flowed away freely from the down-stream edge, the hydrostatic head would be zero at this latter point. It might be a fair assumption to conclude that the pressure varied as a straight line between the up-stream and down-stream edge,* which would give an equivalent pressure over the entire base of one-half the hydrostatic head assumed acting at that point.

3. The third case might be represented by that which would be intermediate between cases 1 and 2; in other words, where there was easy access to the foundation, but not such easy access from it. Under these conditions the pressure at the heel would be assumed equal to the hydrostatic head, while at the toe it would be equal to that pressure represented by the head of the issuing stream.

It therefore becomes a question for the engineer to decide, from a knowledge of the condition of the foundation, as to what degree of entering water and consequent

* Cf. Proceedings Am. Soc. C.E. for May, 1915, "Experiments on Uplift." These, however, are upon too small a scale to yield conclusions other than those applying to the experiments themselves.

uplift may exist, and to provide for it accordingly. It is just because the matter is based upon judgment that such a diversity of opinion prevails.

Generally it is quite likely that none of the above conditions will strictly apply, but rather varied combinations of them, so that it becomes difficult to conclude how to dispose of this question.

Some engineers demand that the structure shall be designed for the full static head acting over the entire base, while others advise that no allowance whatsoever be made, but it is generally conceded that *some* dams, dependent on the kind of their foundations, need provision for uplift.

As an example of the former, there may be cited the dam at Marklissa, Prussia, over the Queis,* while the New Croton Dam is an example of a very important structure of this type where such provision was absolutely eliminated in the design.

There are several ways in which upward pressure may be cared for: First, by adding a sufficient section to the dam to offset the upward pressure, and second, by providing drainage wells and galleries to intercept all entering water, carrying it away through a discharge gallery, or conduit, to the lower side of the dam, and at the same time by carefully providing for as impervious an up-stream face as possible.

In the foundation an adequate cut-off, of width and depth determined by examination of conditions disclosed during the progress of foundation excavation, is often

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* Trans. Am. Soc. C.E., Vol. LXXV.

advisable. The exploratory borings should usually indicate beforehand this necessity, so that the final extent to which a cut-off trench is taken remains to be decided during its actual excavation. Borings may be extended from its faces and bottom to reach seams and pockets to be filled by grouting under pressure before the concrete of the cut-off trench is placed.

The drainage wells, slightly inclined to the vertical, and the cut-off are placed as near as consistent to the up-stream face of the dam, and galleries are built longitudinally to the up-stream face. While the wells and galleries may be nearly completely effective in intercepting percolation, they cannot be considered absolutely so, and consequently may allow some water to get down-stream. Such seepage would then result in upward pressure down-stream from the wells and galleries, and if the water had connection in any way with the reservoir, pressure on a joint due to the full static head might result. These last remarks apply, but with less force, perhaps, to the foundation cut-off.

The theory of this intercepting drainage system is that any water having gotten into the dam due to the static pressure acting on faults or cracks in the comparatively more impervious up-stream face, will be caught and prevented from going any further into the structure.

It should not be assumed that because of an impervious up-stream face, and because of drains and cut-offs, no water reaches the body of the dam below the latter, for there may be construction joints, and contraction cracks in the face, and in places the cut-offs and the down-

stream portion may be less pervious than the up-stream part, as a consequence of which uplift may exist in the latter.

Additional means of protection should be provided in the form of drainage channels to lead the water away from the down-stream portion of both the foundation and the body of the dam.

As an example of intercepting drains, at the Cataract Dam, which furnishes the water supply for Sydney, Australia, "the upper face of masonry was built with special care to a depth of 2 or 3 feet, and this alone is relied upon to prevent seepage. The rest of the dam is built of good, though more pervious, masonry, and throughout the whole were placed 6-inch rectangular conduits filled with broken stone parallel to and about 6 feet back from the up-stream face. These are collected into 6-inch earthenware pipes, laid at right angles to the longitudinal axis of the dam, with exits on the down-stream face." * (Cf. cross-section, Olive Bridge Dam, page 96.)

These systems of drainage naturally tend to eliminate upward pressure and consequently increase the stability of the dam, and would seem justifiable in the case of all important structures.

In small dams drainage wells are not so easily provided and the protection is relatively less complete, because there is a certain minimum distance from the upper face within which the drains cannot well be extended.

With the correction for upward pressure applied in the form of increased section, the water entering the dam is wasted, which is a considerable item of cost, while in

* Mr. Allen Hazen, Trans. Am. Soc. C.E., Vol. LXXV, p. 154.

addition the cost of the structure is increased by the added masonry. With collecting galleries both of these items may be partially eliminated. The cut-off wall in the foundation will often prove an economy in this respect.

The question of cost may become a very important one, as in the case of hydro-electric developments where additional cross-section may mean such an increase in the cost as to cause the abandonment of the project.

TABLE I

WIDTH OF BASE OF DAM GIVEN IN FEET FOR VARYING HEIGHTS

Height of dam in feet.....	5	10	30	60	100	250
Water pressure only (horizontal).....	3.2	6.4	19	40	65	161
Uplift as described below....	4.2	8.4	25	51	84	211

The effect of uplift in its tendency to increase the mass of masonry is shown in Table I, for six triangular dam cross-sections, where the upward pressure is assumed equal to the static head at the heel, and diminishing as a straight line to zero at the toe. For convenience, also the width at the top is taken equal to zero.*

Table II and Fig. 1 represent five cross-sections of dams, four of which were directly designed for comparative purposes, showing the effect, not only of uplift, but of ice pressure, upon the top and bottom dimensions, superelevation necessary, as well as the comparative volumes resulting. These differ from those previously cited, in that the full hydrostatic heads for dams sustaining different heads are there employed for "uplift" comparison,

* Mr. W. J. Douglas in Trans. Am. Soc. C.E., Vol. LXXV, p. 207.

TABLE II

DIMENSIONS, CONDITIONS, ETC., FOR FIVE CROSS-SECTIONS OF DAMS.

Cross-sections.	Conditions of loading. (Masonry, 140 pounds per cu.ft.)	Top width, in feet.	Super-elevation above normal reservoir surface, in feet.	Base, in feet.	Area, in square feet.	Percentage of excess of area over the area of *D*.
Austin, Pa.	.....	2.5	2.5	30	840±	0±
A.....	Ice,* uplift, and horizontal water thrust.....	10.0	10.0	41.5	1475	76
B†	Uplift and horizontal water thrust.	6.0	5.0	40.0	992	19
C†.....	Uplift and horizontal water thrust.	10.0	10.0	39.5	999	20
D.....	Horizontal water thrust.....	5.0	5.0	33.1	836	0

* 21,500 pounds per lin. ft. of dam.

† *B* and *C* are subjected to the same conditions of loading. Flood level at +2.5 ft.

The resultant on any base or horizontal joint of masonry is at the down-stream "middle third" point.

while but one depth of reservoir is employed in the latter set of comparisons.

The uplift intensity is assumed as varying uniformly from a maximum at the heel to zero at the toe and the uplift is considered as acting over only a portion of the area of the joint. This is cared for by assuming only a portion of the full hydrostatic head as acting at the up-stream end of the joint considered. Two-thirds of the full up-stream head is used.

Extent and Distribution.—Upward pressure cannot act over the entire area of a joint or base, or the dam would be floating. Total pressure on the parts of the joint in contact must equal the difference between the weight of dam and uplift, and be less than the crushing strength

of the material or not more than 500 pounds per square inch. "The assumption that only one-third of the rock

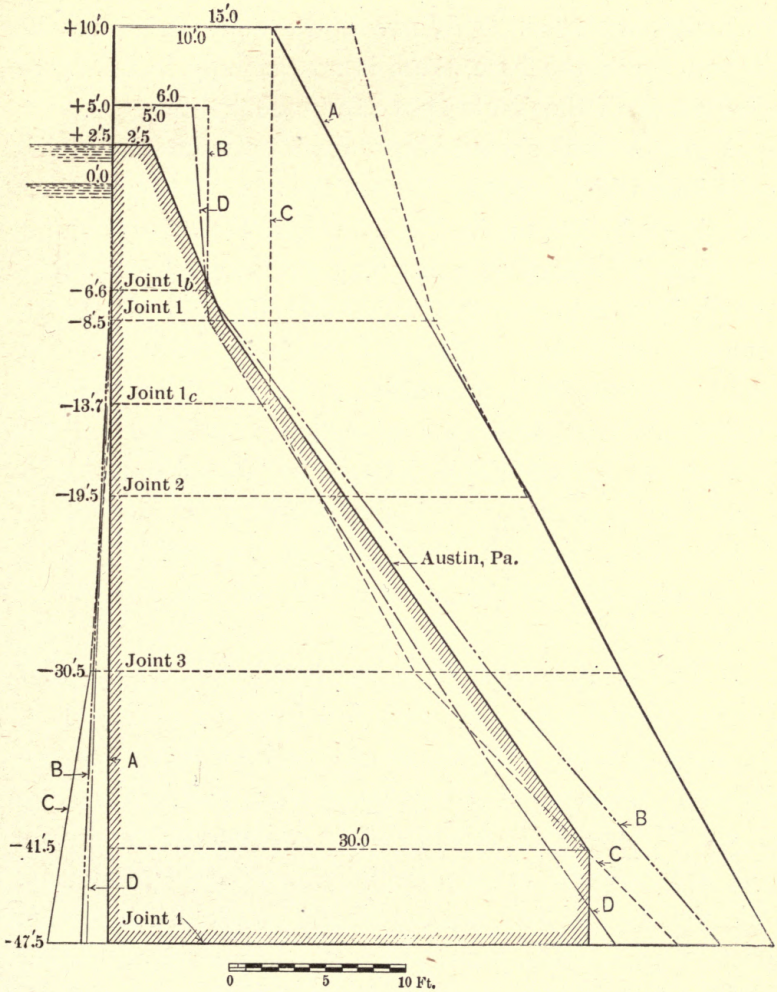


FIG. 1.

faces are in direct contact gives pressure as great as probably occur, and leads to the conservative assumption of

upward pressure over two-thirds of the base, and varying according to the resistance losses in passing through the rock or masonry." Hence a uniformly varying intensity is usually employed. "Where upward pressure must be allowed for in the base, there is no economy in failing to allow for it in the joints above the base."

A distinction should be drawn between the uplift conditions which may be encountered in the foundations and those higher up in the dam, and the foregoing assumptions in regard to uplift may be modified for special cases. For example, a trapezoidal (instead of a triangular) distribution of intensity due to uplift may be found advisable for a foundation. In cases where intercepting drainage wells are provided in the body of the masonry, as for the Olive Bridge and Kensico Dams, it would be reasonable to assume a triangular disposition of intensities, with the maximum at the heel as before, but running out to zero at or a little beyond the line of wells.

It has been suggested as reasonable to assume that if water is to be properly excluded from a masonry dam, the same general methods should be applied as are employed in waterproofing any foundation, such as the use of several layers of tarred felt, a waterproofing surface coat of some kind, or by pouring *wet concrete* continuously. Objection to the first method would be that such foreign substance, in layers, would form a plane of cleavage that would defeat its very purpose by providing a weak, horizontal joint.

It is known that "joints between two successive days' work in concrete may become planes of entry for the water."

Furthermore, when mortar is used in ashlar or rubble masonry it must be dry enough to handle, and consequently it lacks the wet consistency necessary to make it waterproof, thus permitting water to enter at the joints.

As a general proposition, then, upward pressure and ice thrust may be said to be more or less dependent upon local conditions, as to the extent to which they should be considered in any given case.

Three cases of failure, undoubtedly due in part to upward pressure for which no allowance was made in the design, may be instanced, but at the same time it should be pointed out that all of these dams were established on poor foundations.

	Maximum Height.	Built.	Failed.
Bouzey Dam, France.....	72	1878–81	April 27, 1895
Austin Dam, Texas.....	68	1891–92	April 7, 1900
Austin Dam, Pa.....	50	1909–10	Sept. 30, 1911

In the Bouzey Dam the foundation was on fissured red sandstone. and quite permeable, and the excavation was carried down to only a fairly good bottom, and by no means to solid rock. The foundation of the Austin, Texas, dam was located partly over a fault 75 feet wide, filled with adobe with occasional streaks of red clay, nor was the foundation trench excavated deep enough, while the protection on the downstream side was insufficient. The Austin, Pa., dam was founded on sandstone, underlaid by shale having fissures filled with clay, sand, and gravel.

Part II—Ice Thrust

It is realized that, in countries where low temperatures prevail in winter, the pressure of expanding ice against the face of a dam at the level of the water surface may become, before crushing of the ice takes place, a very tangible force tending to destroy the equilibrium of forces and thus overturn the structure. This expansion will occur when the ice is formed under low temperatures and when higher temperatures later prevail.

In addition to this, the ice may deliver a considerable blow or impact when it has formed into floes and the wind carries it down to the face of the dam. In this latter connection, however, it is well to remember that under the force of the wind the jagged points of the ice floes would first come into contact with the dam, and these would be broken off. But under any conditions the expansive force of the ice will, without question, be the more important consideration.

It is to guard against these forces which produce an additional tendency to destroy the equilibrium of the structure, that ice thrust is considered.

Where the reservoir has sloping sides it would seem reasonable to assume that the expanding ice would tend to slide up the shores, and, as the face of the dam is only a small part of the shore line, that there would be comparatively little force exerted against it. If the walls of the reservoir were vertical, however, this would not apply.

In very cold climates it may not be wise nor safe to assume that the ice may be kept clear of the dam by means

of maintaining an open trench next to the upstream face, for on the contrary the full pressure may be exerted.

In the Cross River Dam and in the Croton Falls Dam, recently erected on the watershed of the Croton River near the new Croton Reservoir, and with no inhabitants immediately below them, besides allowing for upward pressure, ice thrust was figured at 24,000 pounds per linear foot for the former and 30,000 pounds per linear foot for the latter.

However, in this connection, it should be borne in mind that reservoirs for domestic supply are usually drawn down during the period when ice prevails, and that a point of application of the ice thrust thus results much lower than might be assumed, which the structure is better able to resist. But if a storage reservoir is to be kept at high level during the ice season, full pressure should be considered acting at the top.

The late C. L. Harrison * concluded that under the following conditions it was not necessary to provide for ice pressure:

"(1) For the ordinary storage reservoir with sloping banks, in climates where the maximum thickness of ice is 6 inches or less—for dams with southern exposure this limit may be placed as high as 1 foot.

"(2) For reservoirs which are filled during the flood season and from which all the stored water is drawn off each year during the low-water season. This would include even the large reservoirs on the head-waters of the Mississippi River, where the ice has a thickness of more than 4 feet, and the atmospheric temperatures reach 50° below zero.

* Trans. Am. Soc. C.E., Vol. LXXV, p. 219.

"(3) For storage reservoirs where the water will be drawn off each year during the winter to a level where the dam is strong enough to resist the ice pressure.

"(4) For reservoirs where the contour of the ground at the high-water level is such that the expansive force of the ice will not reach the dam."

Ice thrust has a greater influence on the thickness of a dam than the flood water level, down to about 110 feet below the flow line in the Olive Bridge Dam, at which point the 10-foot flood level begins to require a wider base; and in the Kensico Dam, with a flood level of 5 feet, the change is about 210 feet below, indicating that, for dams of moderate height, ice pressure has a very great influence. (Cf. 5 Profiles of Fig. 1, page 13.)

In the Olive Bridge Dam it was assumed that clear block ice 1 foot thick might be expected to form at the surface, and expand so as to exert its full crushing strength of about 47,000 pounds per linear foot of dam, and this figure was used in the Wachusett Dam; 42,000 pounds per linear foot was recommended in the Quaker Bridge Dam, and 30,000 pounds per linear foot in the Croton Falls Dam, and 24,000 in Cross River Dam, while in the Design of the New Croton Dam ice thrust was disregarded.

In a discussion regarding ice pressure, before the Canadian Society of Civil Engineers, in December, 1891, agreement seemed to have been reached on two points: That thrust from ice less than 3 inches thick can be disregarded, and that the thrust can safely be taken at the crushing strength of ice. The "Engineering News" of January 12, 1893, and of April 5, 1894, records the compressive

strength of ice ranging all the way from 100 to 1000 pounds per square inch; 21,500 pounds per linear foot of dam, with ice at 3 inches thick would be equivalent to about 600 pounds resistance per square inch. The value of 47,000 pounds above noted is equivalent to about 650 pounds per square inch for ice 6 inches thick.

The uncertainties of the problem are thus increased by the lack of exact and more extended data bearing upon the subject of ice pressures. It is, therefore, largely a matter of judgment as to what should be used as an ice thrust in any given case, as has been indicated by the foregoing discussion.

CHAPTER II

PRELIMINARY CONSIDERATIONS

THE studies involved in the determination of a gravity cross-section demand an investigation along two general lines:

First, the direct calculation to fix the most economical cross-section under the imposed conditions, and

Second, studies in the comparison of cross-sections ranging between this one, which may be called the minimum, and one of an existing masonry dam, where the conditions and responsibility are practically the same as those under consideration.

Before undertaking such an analysis, however, it will be advisable to consider the manner in which water pressure is exerted against a submerged surface; its amount; the method of determining the point of application of the resultant; the assumed distribution of pressure in a masonry joint; and finally, the action of the forces in and upon the structure.

It may be stated as a general proposition that water pressure acts in all directions against a submerged object and that it depends for its value merely upon the "head," or depth of the center of gravity of the figure below the free surface of the liquid. In consequence of this principle it may be shown that the total normal pressure is represented by

$$P = \gamma A h, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where P = the total normal pressure;

γ = the weight of a unit volume of water;

A = the total area; and

h = the vertical distance of the figure's center of gravity beneath the free surface of water.

The demonstration * may be made by considering the surface to be divided into an infinite number of parts; the total pressure on each one of these elements, depending only upon the weight of water resting upon it, may be written,

$$p = \gamma a h_1, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

in which p = the total normal pressure on the differential area;

a = the differential area;

h_1 = the head on a (practically constant over the differential area).

If, therefore, we take the sum of the pressures on all of these small areas, we shall obtain the previous equation, which is perfectly general and applies to any surface. In the case of a vertical, rectangular strip of the back of a dam, the application of the formula will give a total pressure of

$$P = \gamma b \int_0^H x dx = \frac{\gamma b H^2}{2} \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where b is the constant breadth of the strip, usually taken as one unit, x is the variable, and H is the total height of the rectangle.

* See Merriman's "Hydraulics."

The point on the submerged surface at which this resultant pressure acts may be determined by assuming for an axis the horizontal line in which the surface of the water cuts the plane of the back of the dam, taking the moment of inertia of the surface about this axis, and dividing the result by the *static* moment of the surface with reference to the same axis. Applying this to the strip referred to above, there will result,

$$I=\frac{bH^3}{12}, \text{ the moment of inertia of rectangular strip,}$$

and

$$I_1=I+A\left(\frac{H}{2}\right)^2=\frac{bH^3}{12}+\frac{bH^3}{4}=\frac{bH^3}{3},$$

which is the moment of inertia with regard to the assumed axis.

$$I_s=A\cdot\frac{H}{2}=\frac{bH^2}{2},$$

is the static moment of the surface about the same axis, hence,

$$y=\frac{I_1}{I_s}=\frac{2H}{3} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

is the distance of the center of pressure from the surface of the water.

In the investigation of the distribution of pressure in a masonry joint subjected to external forces, the material is assumed to be rigid, though in reality it is to a certain

degree elastic. This elasticity gives the distribution of stress an indeterminate law, so that neither the direction nor the intensity is actually known at any point. It is certain, however, that the intensity must be zero at the edges, although it may increase with great rapidity to higher values very near the limits of the joint. Investigations have been made within the past few years to obtain more exact information as to this distribution of stress, but so far the results are not completely satisfactory. Reference will be made to this matter in Chapter VIII.

Inasmuch as the exact law of stress variation is not known, one of uniform variation of normal stress has been assumed in all practical treatments of masonry joints.

Fig. 2 represents the simplest case, in which the pressure is assumed to be uniformly distributed over the joint $a\ b$, with the constant intensity p; it might be taken as representing any horizontal joint with a superimposed load acting at its center.

To express this condition of uniform stress algebraically, l may be assumed to be the length of the joint from a to b, while the breadth, perpendicular to the plane of the paper, is taken as unity. The area of the joint will then be l, whence,

$$W = p\,l, \quad \ldots\ldots\ldots \quad (5)$$

or

$$p = \frac{W}{l}, \quad \ldots\ldots\ldots \quad (6)$$

which is the formula for a condition of uniform intensity of stress over the entire joint.

It may be observed here that this pressure is uniform only because the total load represented by W, acts at the center of the joint, and that when the point of application is changed to some other position, there will be an increased stress in that direction toward which the load has been moved, and a corresponding decrease in the opposite direction.

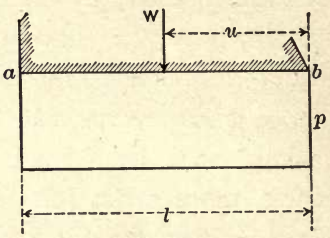


FIG. 2.

It will be necessary therefore, to consider this variation of pressure in eccentrically loaded joints and also the manner in which the eccentricity in the case of a dam is produced.

If $a\,b$ be any plane, horizontal joint in the dam at the distance H below the surface, OY the water surface, and ϕ the angle that the back makes with the vertical, then the total pressure on the back acting at a point one-third the distance up from the joint, will be

$$F' = \frac{\gamma H^2}{2} \sec \phi.$$

Combining this force with the weight of masonry W above the joint acting through the center of gravity of the section, the resultant R will intersect it at some point as e, on $a\,b$, other than the center of figure, called the center of resistance, and it is evident that with a variation of F' and W it may occupy any position along the joint.

Fig. 3, showing only the vertical component, exhibits such a case, where compression exists over the entire joint

as in Fig. 2, but where the center of pressure is not at the center of figure

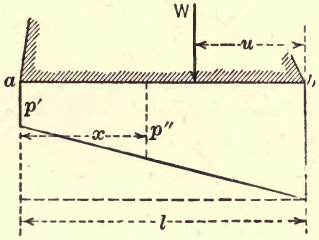


FIG. 3.

If the intensity of pressure at b may be represented by the vertical line p, and the intensity of pressure at a by the line p', then, since by the assumption the pressure varies uniformly over the entire joint, the vertical, p'', at any point, included between the horizontal $a\,b$ and the line joining the extremities of p and p' will indicate the intensity of pressure at that point, while the area of the trapezoid will represent the total pressure on the joint.

The former may be expressed algebraically thus:

$$p'' = p' + (p - p')\frac{x}{l} \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

and the latter by,

$$\frac{(p + p')}{2}\,l, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

The determination of the maximum and minimum pressure p and p' may be made as follows:

Since the static moment of the rectangle $p\,l$ about a point $\frac{1}{3}l$ from p' is the same as the static moment of the trapezoid about the same point, because the moment of the triangle $p-p'$, l about that point is zero, that being the center of gravity of the triangle, there will result by taking moments

$$W(\tfrac{2}{3}l - u) = \frac{pl^2}{6}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

whence,

$$p=\frac{2W}{l}\left(2-\frac{3u}{l}\right), \quad . \quad . \quad . \quad . \quad . \quad (10)$$

which is an expression for the intensity of pressure at the point b, on the joint ab. To solve for the value of p', the intensity of the pressure at the point a, in a similar manner we may take moments about a point $\frac{1}{3}l$ from b, whence,

$$W\left(u-\frac{l}{3}\right)=\frac{p'l^2}{6}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

or,

$$p'=\frac{2W}{l}\left(\frac{3u}{l}-1\right). \quad . \quad . \quad . \quad . \quad . \quad (12)$$

When p' becomes zero, the trapezoid reduces to a triangle as shown in Fig. 4, with its center of gravity at a distance from b equal to $\frac{1}{3}l$, and, since the center of pressure of W must lie vertically above the center of gravity of the triangle graphically representing the variation of pressure over the joint, we shall have, $p'=0$, $u=\frac{1}{3}l$, and Eq. (9) reducing to

$$\frac{Wl}{3}=\frac{pl^2}{6}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

whence,

$$p=\frac{2W}{l}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

That is to say, the maximum pressure p for this condition is twice the value as obtained from Eq. (6).

In Fig. 5 is represented a case in which tension exists over a portion of the joint. p' is here negative.

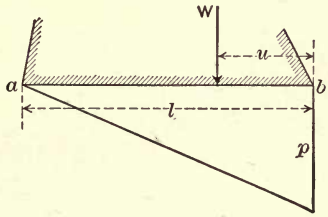


FIG. 4.

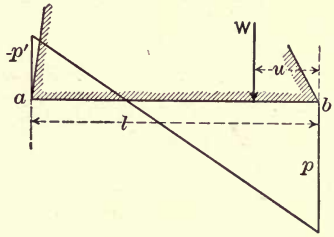


FIG. 5.

Although both masonry and the best hydraulic cement mortar have considerable tensile strength, running up to several hundred pounds per square inch in tests, the latter, together with the continued adhesion of the mortar to the aggregate in concrete, when used, is of uncertain value in this connection. The tensile strength is therefore always neglected in considering the stability of masonry dams or other similar structures, and is an omission which is the more justifiable since it leads to an error on the side of safety.

In the case represented by Fig. 5, the triangle, whose base is $3u$, and altitude p, is therefore alone considered, and by taking moments about b, there will result,

$$Wu = 3u\frac{p}{2}u \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

whence,

$$p = \frac{2W}{3u}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

If it is desirable to know what the tension in the joint

is, it may be determined from Eq. (12). As $\frac{3u}{l} < 1.0$, the resulting value is negative, thus denoting a tension by that equation.

The pressures at a and b may also be determined as follows: Decomposing the resultant acting on any joint into its vertical and horizontal components, V will represent the total normal or vertical pressure, equal to W, the weight of masonry above the joint, *plus* the vertical component of the thrust from the water. The horizontal component of the resultant is disregarded, as its effect upon the joint is more or less indeterminate, and since too, it is assumed to be neutralized by the friction acting in the joint.

The vertical component V, acting through the point of application of the resultant R in the joint, is therefore the factor producing the difference in pressure between a and b, or the uniformly varying stress.

Assume that at the center of the joint, which is not necessarily vertically below the center of gravity of the mass above, two forces equal and opposite to each other, and of the same value V, are applied normal to the joint. The effect of each is to neutralize the other, but if we consider, apart from the other forces, the one acting downward, since it is applied at the center of figure it will produce a uniform stress p over the joint equal to $\frac{V}{l}$.

The two remaining and equal forces V and V, one acting downward at the point of application of R, and the other upward at the center, form a couple whose lever arm is v, and the moment of which is therefore $V \times v$.

This moment produces a uniformly varying stress over the joint, increasing the intensity at b and decreasing it at a by an equal amount.

To determine its value we have but to consider the following:

$$M = Vv. \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

the moment caused by the couple and producing the varying stress. Also,

$$M = \frac{kI}{d_1}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

where k is the intensity of stress at the maximum distance from the neutral axis; I, the moment of inertia of the section about such an axis; and d_1 the normal distance from the neutral axis to that point where k exists.

Since the neutral axis passes through the center of figure of the joint, the value of d_1 is half the length of the joint, while I, the moment of inertia, equals $\frac{1}{12}l^3$, if we consider a horizontal section in the plane of the joint $a\,b$ extending back from the plane of the paper one unit's distance. Hence,

$$M = Vv = \frac{kI}{d_1} = \frac{k\,\frac{1}{12}l^3}{\frac{1}{2}l} = \frac{kl^2}{6}, \quad . \quad . \quad . \quad . \quad (19)$$

or,

$$k = \frac{6\,Vv}{l^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

Here k represents the stress that must be added to the uniform stress $\frac{V}{l}$ to find the intensity of pressure at the

toe b and the amount which must be subtracted from $\frac{V}{l}$ to arrive at the intensity at the heel a. It is expressed in pounds per square inch, but if the distances are measured in feet and the forces in pounds, k will be designated in pounds per square foot.

While it is customary to consider only the normal component of the resultant pressure acting in a horizontal joint and to assume it to vary uniformly, this is probably correct only for horizontal joints in rectangular walls vertically loaded and not subjected to lateral pressures. It will be shown later that the maximum stresses exist at or near the down-stream face, and act in a direction parallel to and on planes normal to that face. The fact also that acute edges do not crack off in the inclined faces of dams is in itself a partial confirmation of the statement.

Under these circumstances then, the maximum normal pressure in a horizontal joint must be much less than the actual maximum pressure in the dam, and it has been assumed to bear the ratio to the latter of about 9 to 13.

CHAPTER III—PART I

DEVELOPMENT OF FORMULÆ FOR DESIGN

SIX series of formulæ, designated by the letters A, B, C, D, E, and F, will now be presented, in each of which a given set of conditions with respect to the external forces will be involved; but as the method of procedure is practically the same for all cases, only series A will be developed here.

The following nomenclature will be employed:

L = the width of the top of the dam cross-section;

l = length of a horizontal joint of masonry, *to be determined;*

l_0 = *known* length of the joint next above joint of length l;

h = depth of a course of masonry (vertical distance between l_0 and l);

P = line of pressure, reservoir full;

P' = line of pressure, reservoir empty;

u = distance from front edge of the joint l to the point of intersection of P with the joint l, measured parallel to joint l;

y = distance from back edge of the joint l to the point of intersection of P' with the joint l, measured parallel to joint l;

y_0 = distance from back edge of the joint l_0 to the point of intersection of P' with the joint l_0, measured parallel to joint l_0;

v = distance between P and P' at the joint l, measured parallel to joint l;

γ = weight in pounds of a cubic foot of water (62.5);

γ' = weight in pounds of a cubic foot of mud (75–90);

Δ = ratio of unit weight of masonry to unit weight of water (often assumed as $\frac{7}{3}$);

$\Delta\gamma$ = weight in pounds of a cubic foot of masonry;

H = head of water on joint l (vertical distance of joint l below water surface);

H' = depth of earth back fill over joint l on front;

H_1 = head of water on joint l when ice acts at surface of water;

$H-H_1$ = rise of water level, due to flood, wave, etc., above normal level for full reservoir;

h_1 = head of water above mud level (liquid mud of weight γ');

h_2 = head of liquid mud on joint l, on back;

a = vertical distance from the top of the dam to the surface of water (flood);

a_1 = vertical distance from the top of the dam to the surface of water when ice is considered (a_1 generally exceeds a);

b = vertical distance from water surface to top of dam when dam is overtopped;

c = ratio of upward thrust intensity, due to hydrostatic head H (or H_1, or h_1+h_2), assumed to act at heel of joint l (usually assumed as $\frac{2}{3}$);

$T\gamma$ = horizontal ice thrust at water surface in pounds (47,000);

(The value here given, for example, was used in studies for design. Our present lack of exact data in regard to ice pressures prevents more than a speculation from being made as to a definite value to be assigned in any case);

$D\gamma$ = horizontal dynamic thrust of water in pounds;*

$E\gamma$ = thrust of earth back fill in pounds (on front);

$W_v\gamma$ = vertical pressure on inclined upstream face above joint l, in pounds;

A_0 = total area of cross-section of dam above joint l_0;

A = total area of cross-section of dam above joint l.

t = batter of upstream face for vertical distance h;

s = distance of line of action of $W_v\gamma$ from upstream edge of joint l, measured parallel to joint l;

δ = angle that $E\gamma$ makes with horizontal;

α = angle of slope of downstream face of dam with horizontal;

β = angle R makes with the vertical;

p = maximum allowable pressure intensity at toe (in pounds per square foot);

q = maximum allowable pressure intensity at heel (in pounds per square foot) (p is assumed less than q) p and q may be used to signify the calculated, existent pressure intensities

* Determined, as in the case of the overfall dam, by the probable velocity of flow against the dam.

corresponding to P and P' respectively, for the joint l.

f = the coefficient of friction for masonry on masonry (usually 0.6 to 0.75);

S = the shearing resistance of the masonry per square unit;

$F = \frac{\gamma H^2}{2}$ = the horizontal static thrust of the water in pounds;

$M = \frac{\gamma H^3}{6}$ = the moment of F about any point in the joint l;

$W = A\Delta\gamma$ = the total weight, in pounds, of masonry resting on the joint l;

$W_0 = A_0\Delta\gamma$ = the total weight, in pounds, of masonry resting on the joint l_0;

R = the resultant of F and W;

R' = the resultant of the reactions;

$\frac{cHl\gamma}{2}$ = upward thrust of water on base l.

In the figures, hydrostatic pressures are indicated by triangular and trapezoidal areas included within dotted lines, while ice pressure is shown to contrast H_1 with H.

As before, if a unit length of one foot of dam be considered, the letters T, D, E, W_0, A, A_0, and H^2 will signify volumes.

It will be observed that, where possible, the several equations have been cleared of the term $\Delta\gamma$, thereby simplifying actual calculations.

In the above table c, in a manner, may be considered to provide for an assumption of a certain proportion of the

joint's area being subjected to upward water pressure; and the distribution, as evidenced by $cHl\gamma/2$, varying from a maximum intensity at the heel to zero intensity at the toe, is assumed in view of the fact that the tendency to open the joint would begin at the heel while a zero intensity of upward pressure at the toe would presuppose an opening with consequent flow at that point. As the dam would then be failing in its chief function, i.e., to retain water, this flow is not considered to exceed a slight seepage.

In general four ways are recognized in which a masonry dam may fail:

1. By overturning about the edge of any joint, due to the line of action of the resultant passing beyond the limits of stability.

2. By the crushing of the masonry or foundation because of excessive pressure.

3. By the shearing or sliding on the foundation or any joint, due to the horizontal thrust exceeding the shearing and frictional stability of the material.

4. By the rupture of any joint due to tension in it.

An unsatisfactory foundation might also be mentioned as possibly leading to failure, and in view of this, the footing upon which the dam rests should always be most carefully scrutinized.

To preclude failure from any of the above mentioned causes, it is the practice to design the cross-section of the dam with the following conditions imposed:

1. The lines of pressure, both for the reservoir full and empty, must not pass outside the middle third of any horizontal joint.

2. The maximum normal working pressure on any

horizontal joint must never exceed certain prescribed limits, either in the masonry itself or in the foundation.

3. The coefficient of friction in any plane horizontal joint, or between the dam and its foundation, must not be less than the tangent of the angle which the resultant makes with a vertical.

As may be seen by referring to the figures showing the distribution of pressure on a joint, when the resultant lies within the middle third, tension can exist in no part of it, nor can the safety factor be less than two, if we neglect to consider the upward pressure of water percolating through any of the joints or beneath the dam.

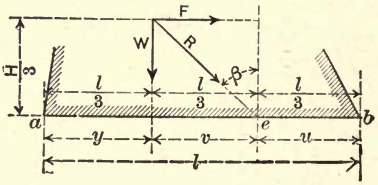


FIG. 6.

To illustrate the conditions that exist and to derive the value of the safety factor when the resultant cuts the joint at the extremity of the middle third, we may take the case as shown in Fig. 6. Resolving R into its horizontal and vertical components, and taking moments about the center of resistance e, the following equation is obtained:

$$F\frac{H}{3}=W\frac{l}{3}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

where F is the horizontal component of the thrust from the water behind the dam, acting at a point $\frac{1}{3}H$ above

the plane of the joint, while W is the vertical component of the resultant, and as such, includes not only the weight of the masonry, but the vertical component of the thrust from the water as well, provided the latter is considered as acting normal to the back of the dam.

For the dam to be on the point of rotating about b, the downstream edge of the joint, it is obvious that the resultant R must pass through that point. Under these circumstances, since the lever arm of F is still $\frac{H}{3}$, and the lever arm of W has been increased to twice its former value or $2\left(\frac{l}{3}\right)$, for the above equation to still hold, F must also be increased to twice its former value. This would indicate that when R acts through the point e, the value of H is only one-half as great as is necessary to produce overturning; or, in other words, that the factor of safety is two as indicated by the ratio of $\frac{(u+v)}{v}$. It should be observed however, that the material near the edge of the joint will crush some time before the resultant has reached it, and that therefore the factor of safety against overturning with R at the limit of the middle third is something less than two.

When, however, the upward pressure of water acting over the joint due to percolation is taken into consideration, the factor of safety will be somewhat modified, as the following demonstration will make clear.

By referring to Fig. 7, it will be seen that, for example, the horizontal water pressure on the back, $\frac{\gamma H^2}{2}$, the uplift,

$\frac{cH\gamma l}{2}$, and the weight of the masonry, W, constitute the forces, with the reaction (not shown) considered as acting on the dam.

The forces tending to overturn the dam about the

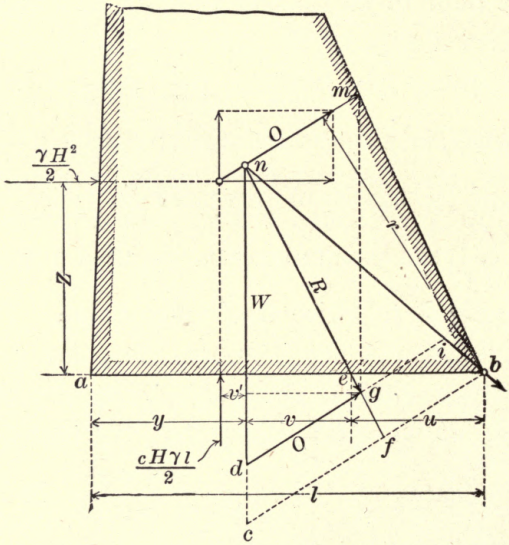


FIG. 7.

toe at b are $\frac{\gamma H^2}{2}$ and $\frac{cH\gamma l}{2}$, the resultant of which is denoted in magnitude and direction by the line, O, at the normal distance, r, from b. The force resisting this tendency is W, the line of action of which is normally distant $(u+v)$ from b. If nm denotes the force, O, in magnitude and direction, and nd, the force, W, then ng will represent, in magnitude and direction, the force, R, or the final resultant of all the forces, which is opposed by the reaction

(for equilibrium to be assured) at the point, e, distant u from b.

The resisting moment about b, then, is $W(u+v)=M_0$ while the overturning moment about the same point is

$$rO=M'=\frac{\gamma H^2}{2}\times z+\frac{cH\gamma l}{2}(u+v+v')=M_1+M_2.$$

Whence, for the ratio of "resisting moment to overturning moment" there may be written:

$$\frac{M_0}{M'}=\frac{M_0}{M_1+M_2}, \quad . \quad . \quad . \quad . \quad . \quad (21a)$$

and for the ratio of the "resultant moment of the vertical components" to the "resultant moment of the horizontal components" of the forces there follows:

$$\frac{M_0-M_2}{M_1}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (21b)$$

These two expressions for the "factor" evidently become equal to each other only when $M_2=0$, or when uplift is ignored; also, when the factor of safety is equal to unity (when $M_0-M_2=M_1$), or at the point of overturning. Therefore, the usual expression, $\frac{u+v}{v}$, will not be the value for the "factor of safety" against overturning, according to Eq. (21a).

To consider the foregoing discussion for the purpose of developing a graphic treatment for determining the value in either case, it must be remembered that, for the dam to

be on the point of overturning, the ratio of the two moments, M_0 and M' of Eq. (21*a*), must equal unity or $M_0 = M_1 + M_2$, and the line of action, R, in Fig. 7, must pass through b. Inasmuch as W is constant, one or all of the other forces may at this stage be considered variable, in order to bring about the above supposititious condition. According to Eq. (21*a*), the distance, r, is constant, and the water pressure on the back and the uplift, therefore, are supposed to be proportionately increased to fulfill this condition, of bringing the line of action, R, through b. This seems reasonable from the fact that the horizontal water thrust cannot be considered to increase without a corresponding increase in the uplift. Therefore, the condition necessary to bring the resultant, R, through b instead of through e, where it actually falls, is that $O = dg$ be increased to di, in Fig. 7. If bc be drawn through b, parallel to O (and, therefore, to dg), the ratio sought follows from the similarity of the triangles, nid and nbc, or, the factor of safety, with respect to resisting moment and overturning moment, is equal to the ratio, $\frac{di}{dg} = \frac{cb}{cf}$.

The foregoing conception, Eq. (21*b*), of the "factor of safety" tacitly assumes that only the horizontal thrust of the water is instrumental in moving the center of pressure from e to b, and that the uplift merely lessens the resisting moment.

As the overturning force to be increased is therefore horizontal, and as the length of the line parallel to the overturning resultant and comprehended between the point, b, and the line of action of the resisting force is divided by its segment (comprehended between the actual

resultant and the same line of action of the resisting force) to get the ratio, or factor against overturning, it at once follows that the division according to Eq. (21*b*) would be $\frac{u+v}{v}$. Eq. (21*a*) seems preferable, or the "factor of safety" $=\frac{cb}{cf}$, as in Fig. 7. The "factor of safety," however, is of doubtful value, due to the certain impossibility of the structure's rotation about the point *b*; but it may be a useful quantity for comparison at times. The expression (21*a*) may give values less than (21*b*) by as much as one-third, in some cases.

As was stated previously, the frictional and shearing resistance of a joint is assumed to withstand the tendency of the horizontal thrust to slide the upper portion over the lower, so that it is quite customary, even though it should be investigated, to neglect it.

For equilibrium in this regard,

$$F \leqq fW + Sl, \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

where F is the horizontal component of the water's thrust, f the coefficient of friction, usually taken between 0.6 and 0.75 for masonry, and S is the shearing resistance per unit of area.

In spite of the fact that S has an appreciable value, and particularly so for monolithic masses of "cyclopean masonry," the value is practically indeterminate, and consequently usually ignored. Numerous attempts have been made however, to write expressions for it, the most rational of which depends upon the trapezoidal law of the distribution of normal stress; but this too is unsatis-

factory from a practical standpoint.* We shall neglect S, therefore, in the previous equation, whence,

$$F \overset{=}{<} fW, \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

which gives at the limit,

$$f = \frac{F}{W} = \tan \beta. \quad . \quad . \quad . \quad . \quad . \quad (24)$$

In every design the imposed conditions for equilibrium result in a cross-section in which the back has very much less of a batter than the front. It may be shown also,* that, as the shear along either face is zero, the greatest intensity of stress will act in a direction parallel to the face at, and near, the edge. Since the horizontal component of the pressure is ignored, this implies that the greatest vertical, or normal working intensity of pressure must be less at the downstream face where the inclination is greater than at the heel, in order that the components parallel to the respective faces shall be approximately equal. This is accomplished by using a smaller vertical normal working stress at the toe than at the heel.

As the up-stream face of a masonry dam is vertical for a considerable distance from the top, and then becomes only slightly inclined to it, it is customary to consider the thrust from the water as acting horizontally. This is the more justifiable since the vertical component of the water resting upon the up-stream face of the dam causes an overturning moment about the center of resistance, opposite in direction to that induced by the horizontal thrust, and hence is an error on the side of safety.

* See Chapter VIII.

It must be evident from the equation of pressure, $p = \gamma ah$, that where this alone governs the resulting theoretical cross-section, it will be triangular in form with the apex at the surface of the water; but where it is intended there shall be no flow over the crest of the dam, it is customary to carry the masonry some distance above the elevation of the water in the reservoir, not only to allow for fluctuations, but because of economic conditions or to provide for a foot or carriage way. The superelevation and the width of top are therefore arbitrarily assumed and should be taken at about $\frac{1}{10}$ the height of the dam, with a minimum width of 5 feet and a maximum superelevation of 20 feet.*

As no equation can be written simultaneously expressing the three conditions of stability, i.e., that the resultant lie within the middle third, that the maximum pressures shall not exceed certain limits, and that the horizontal components shall not cause sliding, it becomes necessary to determine the length of joints (usually taken vertically 10 feet apart for a depth of about 100 feet and increasing to 20 or 30 feet below), by the aid of that equation involving the limiting conditions which are known to apply, in order that the cross-section be a minimum, and then to test the joint, if necessary, by the other two. Generally speaking the third condition will be found to hold if the joint has been designed in accordance with the other two.

* Mr. William P. Creager, in Proc. Am. Soc. of C.E., for Nov., 1915, "The Economical Top Width of Non-overflow Dams," shows this width to lie between 10% and 17% of the height, according to design assumptions and concludes that exceptionally wide tops may be used, there being but slight economy in adopting narrow tops.

Considering Fig. 6, in which l is the length of joint, it is seen to be divided into three parts, u, v, and y, and from this what may be called the fundamental equation of the entire design can be written.

$$l = u + v + y. \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

If M represents the overturning moment about e, then we have that at the limit of the middle third,

$$M = F\frac{H}{3} = Wv, \quad . \quad . \quad . \quad . \quad . \quad (26)$$

or,

$$v = \frac{M}{W}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

As the analysis will result in a cross-section polygonal in outline, composed of trapezoids with bases l and l_0 and altitudes h, we may write a general equation,

$$W = W_0 + \left(\frac{l + l_0}{2}\right) h \Delta\gamma, \quad . \quad . \quad . \quad (28)$$

or,

$$A\Delta\gamma = A_0\Delta\gamma + \left(\frac{l + l_0}{2}\right) h \Delta\gamma,$$

whence,

$$A = A_0 + \left(\frac{l + l_0}{2}\right) h, \quad . \quad . \quad . \quad . \quad . \quad (29)$$

and since;

$$\frac{W}{\Delta\gamma} = A_0 + \left(\frac{l + l_0}{2}\right) h,$$

then,

$$v = \frac{\dfrac{M}{\Delta\gamma}}{A_0 + \left(\dfrac{l + l_0}{2}\right) h}, \quad . \quad . \quad . \quad . \quad . \quad (30)$$

which value of v, if substituted in Eq. (25) gives,

$$l = u + \frac{\frac{M}{\Delta\gamma}}{A_0 + \left(\frac{l + l_0}{2}\right)h} + y, \quad . \quad . \quad . \quad (31)$$

The above Eq. (31) is a modification of Eq. (25) and, when proper values have been assigned to u and y, depending upon the existing conditions, is used throughout the entire design in the determination of the length of joints.

In the upper rectangular portion of the dam, where there is an excess of material above that required by the static pressure of the water, it will be found unnecessary to consider failure from crushing, as the maximum normal pressures are well below the allowed working pressure, and consequently the depth at which the section ceases to be rectangular will be fixed by the fact that the resultant may not pass outside the middle third. The algebraic expression for this condition is,

$$u \geqq \tfrac{1}{3}\, l \text{ for reservoir full,} \quad . \quad . \quad . \quad . \quad (32)$$

$$y \geqq \tfrac{1}{3}\, l \text{ for reservoir empty.} \quad . \quad . \quad . \quad (33)$$

Below this rectangular portion, trapezoidal sections will be found. At the base of the rectangle, $l = l_0 = L$, $u = \frac{L}{3}$, and, since the center of gravity of the figure is vertically above the center of the joint, $y = \frac{L}{2}$.

If we wish to determine the depth to which the rectangular portion extends, we may do so by the use of Eq.

(31), which, as shown, must involve the condition that the resultant shall just touch the limit of the middle third, i.e., $u=\frac{L}{3}$. Substituting in Eq. (31), $u=\frac{L}{3}$, $y=\frac{L}{2}$, and remembering that $A_0=o$

$$L=\frac{L}{3}+\frac{\frac{H^3}{6\Delta}}{L(H+a)}+\frac{L}{2},$$

whence, by dividing by L,

$$1=\frac{5}{6}+\frac{H^3}{6\Delta(H+a)L^2}=\frac{H^3}{L^2(H+a)\Delta},$$

or, solving for H,

$$H=\sqrt[3]{\Delta L^2(H+a)}, \quad . \quad . \quad . \quad . \quad (34)$$

If $H=h$ then $a=o$, and Eq. (34) reduces to,

$$H=h=L\sqrt{\Delta}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (35)$$

At this depth the rectangle ceases, the sections become trapezoidal, the back face is still vertical but the front face h', is inclined in order to increase the length of the successive joints and thus maintain the resultant for the reservoir full at the downstream limit of the middle third. For a considerable distance below the rectangular section therefore, Eq. (31) will be used with $u=\frac{l}{3}$ to de-

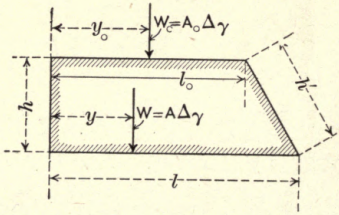


FIG. 8.

termine the length of joint, and the back face will remain vertical, but for each new joint the resultant for the reservoir empty will approach nearer and nearer to the limits of the middle third, until finally $y=\frac{l}{3}$.

It is therefore expedient to compute the value of y under these conditions to learn exactly at what vertical depth or joint this value of y first equals $\frac{l}{3}$.

To do this, moments are taken about the vertical face, for both A_0 and the trapezoid, the latter being found by dividing the trapezoid into a rectangle and a triangle; its value is,

$$\frac{l^2_0 h}{2}+\frac{h}{2}(l-l_0)\left[l_0+\frac{l-l_0}{3}\right]=\frac{h}{6}(l^2+l\,l_0+l^2_0),$$

and hence,

$$Ay=A_0y_0+\frac{h}{6}(l^2+l\,l_0+l^2_0).$$

Substituting the value of A from Eq. (29) in the above, and solving for y there results,

$$y=\frac{A_0y_0+\frac{h}{6}(l^2+l\,l_0+l^2_0)}{A_0+\left(\frac{l+l_0}{2}\right)h}. \quad . \quad . \quad . \quad (36)$$

This gives a value of y to be substituted in Eq. (31) while the value of $u=\frac{1}{3}l$, which has been maintained

since leaving the bottom of the rectangular section, is substituted also. There then results by reduction,

$$l^2+\left(\frac{4A_0}{h}+l_0\right)l=\frac{1}{h}\left(\frac{H^3}{\Delta}+6\,A_0y_0\right)+l^2{}_0, \quad . \quad . \quad (37)$$

which is the equation used in the determination of the length of joint from the foot of the rectangular section down to that joint where Eq. (36) first gives a value of $y=\frac{l}{3}$. At this point the back face must be made to slope, while $u=y=\frac{1}{3}l$ is substituted in Eq. (31) to obtain the following:

$$l^2+\left(\frac{2A_0}{h}+l_0\right)l=\frac{H^3}{\Delta h}, \quad . \quad . \quad . \quad . \quad (38)$$

which will determine the length of the joints.

The second condition will be a factor from here on, for below this section at some point, the intensities of the pressures at the toe will gradually approach and finally equal the allowable limit p_2 and the length of the joint will depend primarily upon this. It is therefore necessary, after each application of Eq. (38) to see if the limiting pressure p at the toe, which is smaller than q, at the heel, has been reached. Its value is derived from the equation $p=\frac{2W}{l}=\frac{2A\Delta\gamma}{l}$, and when the limiting value of p has been realized the value of u thereafter must be derived from,

$$u=\frac{2l}{3}-\frac{pl^2}{6A\Delta\gamma}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (39)$$

in which u, is seen to be dependent upon the normal working pressure p at the toe. (Eq. (39) follows directly from Eq. (9)).

There is some distance below this joint, however, where y still remains equal to $\frac{1}{3}l$, while the value of u is being determined from the above Eq. (39). Under these circumstances, l will be found from the following after substituting the values of $y=\frac{1}{3}l$, u from Eq. (39) and A from Eq. (29), all in Eq. (31).

$$l^2=\gamma\frac{H^3}{p}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (40)$$

This equation will be used until a joint has been reached where the application of $q=\frac{2A\Delta\gamma}{l}$ shows its value to be equal to or greater than that prescribed for q. Here y will be determined by

$$y=\frac{2l}{3}-\frac{ql^2}{6A\Delta\gamma}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (41)$$

in which it is seen to depend on q.

When this point has been reached, u will take its value from $u=\frac{2l}{3}-\frac{pl^2}{6A\Delta\gamma}$, and y from the equation $y=\frac{2l}{3}-\frac{ql^2}{6A\Delta\gamma}$, which must be substituted in Eq. (31) to determine l. This will give after reduction,

$$\left(\frac{p+q}{h\Delta\gamma}-1\right)l^2-\left(\frac{2A_0}{h}+l_0\right)l=\frac{H^3}{\Delta h}. \quad . \quad . \quad . \quad (42)$$

All joints below this point will be found by this last equation.

Summing up, we may say that Eqs. (34), (37), (38), (40), and (42), are the five equations to be used in determining the length of joints from the top down. Strictly speaking Eq. (34) gives the depth at which the rectangular portion ceases, while Eq. (37) gives the length of joints from the base of the rectangle down to where $y=\frac{1}{3}l$; Eq. (38) the length of joints from the point where $y=\frac{1}{3}l$ to where p reaches its limiting value; Eq. (40) the length of joints from the point where p equals its limiting value to where q equals its limiting value and Eq. (42) gives the length of all joints below.

Eqs. (34) and (37) involve the value of y, which is obtained with respect to the vertical back, but when that face begins to slope it is necessary to determine it with regard to the back edge of the joint in question.

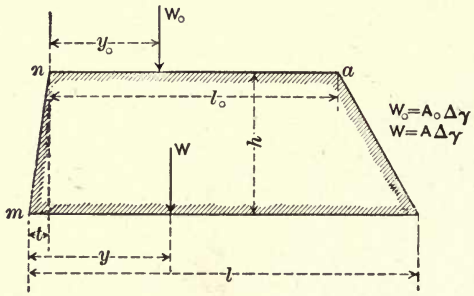


FIG. 9.

In Fig. 9, mn represents the back face of the dam and t is the batter to be determined by taking static moments of A and A_0 about the back edge, m, of the joint.

The trapezoid of the figure is composed of the triangles $ht/2$ and $(l-l_0-t)h/2$ and the rectangle hl_0.

By taking moments about the edge m,

$$Ay = A_0(y_0 + t) + \frac{ht^2}{3} + l_0 h\left(\frac{l_0}{2} + t\right) + \frac{h}{6}(l - l_0 - t)(l + 2l_0 + 2t). \quad (43)$$

For Eqs. (38) and (40) the value of y must be, as before, taken equal to $\frac{1}{3}l$, while A has the usual value of $A_0 + (l + l_0)h/2$. Substituting these in Eq. (43) and reducing:

$$t = \frac{2A_0(l - 3y_0) - hl_0^2}{6A_0 + h(2l_0 + l)}. \quad . \quad . \quad . \quad . \quad (44)$$

For the joints to which Eq. (42) applies the value of y is to be taken from Eq. (41) as was done before. In this case:

$$Ay = \tfrac{2}{3}l\left[A_0 + \left(\frac{l + l_0}{2}\right)h\right] - \frac{ql^2}{6\Delta\gamma}.$$

By substituting this value in the first member of Eq. (43) and reducing:

$$t = \frac{A_0(4l - 6y_0) + l^2\left(h - \frac{q}{\Delta\gamma}\right) + l_0 h(l - l_0)}{6A_0 + h(2l_0 + l)}. \quad . \quad . \quad (45)$$

After the value of l is found by the use of Eqs. (38), (40) or (42), t can at once be determined for the same joint by either Eq. (44) or Eq. (45).

In this manner an entire theoretical cross-section can be determined. It will be noticed that the location of the center of pressure in the middle third of the joint is the governing condition in the upper part of the dam, while the lower portion is fixed by the limiting pressures p and q.

The difficulties preventing the forming of a simple working equation for the entire cross-section arise from the fact that the governing conditions are not introduced simultaneously nor in the same joint.

By taking h of the proper value, a polygonal cross-section may be determined by the preceding formulæ. This cross-section can be then modified by drawing what may be called "mean" lines, straight, broken or curved along the theoretical faces so as to adapt the latter to a practical arrangement and treatment of the joints and facing blocks, which may be of cut stone or concrete.

The conditions which have governed the analysis are essentially those of Rankine, i.e., the center of pressure has in all cases been kept within the middle third of the joint and the greatest intensity of pressure, either at the front face or back, has not been allowed to exceed the limit p or q.

CHAPTER III—PART II

FORMULÆ FOR DESIGN

Series A, B, C, D, E, and F.

As noted earlier six separate series of formulæ for design have been derived and they will be here set forth in suitable form for easy reference and use. As they have been developed by the method just outlined it is unnecessary to follow out the derivation of each series, although there exist some detailed differences in the treatment of each. These details however, would become evident to anyone following the deductions throughout.

Various conditions of "loading," with approved assumptions, such as pressure due to expanding ice at the water surface, upward water pressure on the base, etc., referred to in the table of nomenclature previously given, have been introduced and are specifically stated for each case.

It will be recalled that Eq. (31) is the fundamental expression for finding the length l, of any joint; and, as the several conditions are introduced, that the "M" must in each case signify the total overturning moment and not merely the moment of the static water pressure on the back.

The development of a cross-section, by any one of the following series, may comprise five stages, each stage

representing the introduction of a governing condition. Hence, for each stage there obtains a main equation for finding the length of joint l, each main equation being supplemented by secondary equations for y, u, and t; p and q.

It may be necessary to employ more than one of the series of equations in determining a cross-section.

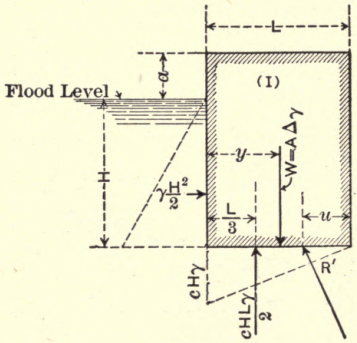


FIG. 10.

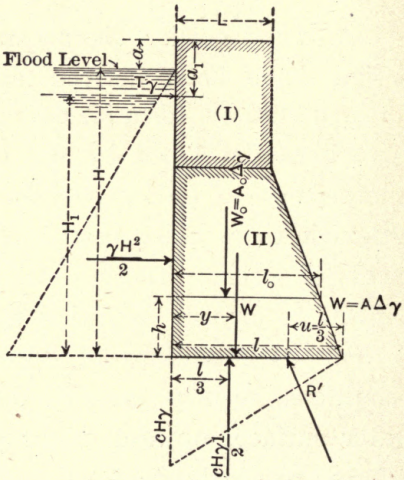


FIG. 11.

For ready reference, the five stages will be set forth and depicted in order as follows:

Stage I.—This stage, it will be remembered, extends from the top of the dam to the joint where the front face commences to batter. It is the rectangular section. $y > \frac{1}{3}L$; $u \geqq \frac{1}{3}L$ (see Fig. 10). Ice pressure is purposely omitted in Fig. 10 to prevent confusion of letters in small space.)

Stage II.—This stage extends from the lower limit of Stage I to the point where the back face commences to batter. $u = \frac{1}{3}l$; $y \geqq \frac{1}{3}l$ (see Fig. 11).

Stage III.—This stage extends from the lower limit of Stage II to the point where the intensity of pressure on the toe has reached the maximum allowable intensity. In this stage $u = \frac{1}{3}l$; $y = \frac{1}{3}l$ (see Fig. 12).

Stage IV. — This stage extends from the lower limit of Stage III to the point where the pressure intensity on the heel has reached the maximum allowable intensity. For this stage $u > \frac{1}{3}l$; $y = \frac{1}{3}l$ (see Fig. 12).

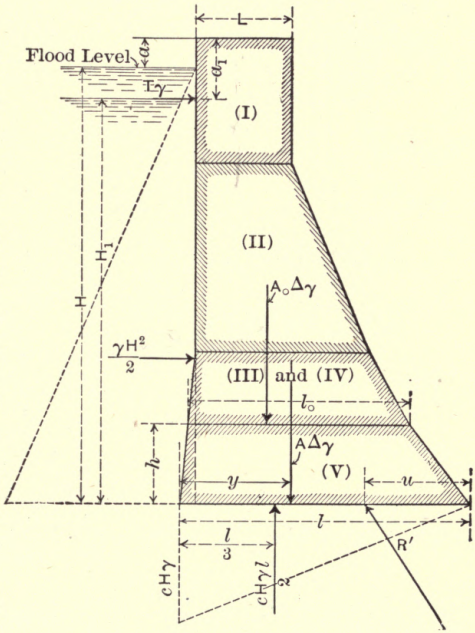


FIG. 12.

Stage V.—In this stage, the limiting intensities of pressure at both toe and heel having been reached, $y > \frac{1}{3}l$; $u > \frac{1}{3}l$.

This stage extends from the lower limit of Stage IV downward. (See. Fig. 12).

The following secondary formulæ, supplementary to the main equations of all series, with substitutions as noted, are arranged in order corresponding with the preceding.

With the condition of hydrostatic upward pressure on the base obtaining, substitute the formulæ in this column in place of those corresponding, as indicated.

Stage I.

$$u \geqq \tfrac{1}{3}L$$

$$y = \tfrac{1}{2}L$$

$$t = 0$$

$$p = \frac{2\Delta\gamma A}{L}\left(2 - \frac{3u}{L}\right)$$

$$p = \gamma\left(\frac{2\Delta A}{L} - cH\right)\left(2 - \frac{3u}{L}\right)$$

$$q = \frac{\Delta\gamma A}{L}$$

Stage II.

$$u = \tfrac{1}{3}l$$

$$y = \frac{A_0 y_0 + (l^2 + l\,l_0 + l_0^2)\dfrac{h}{6}}{A_0 + \left(\dfrac{l + l_0}{2}\right)h}$$

$$t = 0$$

$$p = \frac{2\Delta\gamma A}{l}$$

$$p = \gamma\left(\frac{2\Delta A}{l} - cH\right)$$

$$q = \frac{2\Delta\gamma A}{l}\left(2 - \frac{3y}{l}\right)$$

Stage III.

$$u = \tfrac{1}{3}l$$

$$y = \tfrac{1}{3}l$$

$$t = \frac{2A_0(l - 3y_0) - hl_0^2}{6A_0 + h(2l_0 + l)}$$

$$p = \frac{2\Delta\gamma A}{l}$$

$$p = \gamma\left(\frac{2\Delta A}{l} - cH\right)$$

$$q = \frac{2\Delta\gamma A}{l}$$

Stage IV.

$$u = \tfrac{2}{3}l - \frac{pl^2}{6\Delta\gamma A} \qquad u = \tfrac{2}{3}l - \frac{pl^2}{3\gamma(2\Delta A - cHl)}$$

$$y = \tfrac{1}{3}l$$

$$t = \frac{2A_0(l - 3y_0) - hl_0^2}{6A_0 + h(2l_0 + l)}$$

$$p = \frac{2\Delta\gamma A}{l}\left(2 - \frac{3u}{l}\right) \text{(limiting intensity)} \qquad p = \gamma\left(\frac{2\Delta A}{l} - cH\right)\left(2 - \frac{3u}{l}\right)$$

$$q = \frac{2\Delta\gamma A}{l}$$

Stage V.

$$u = \tfrac{2}{3}l - \frac{pl^2}{6\Delta\gamma A} \qquad u = \tfrac{2}{3}l - \frac{pl^2}{3\gamma(2\Delta A - cHl)}$$

$$y = \tfrac{2}{3}l - \frac{ql^2}{6\Delta\gamma A}$$

$$t = \frac{A_0(4l - 6y_0) + \left(h - \frac{q}{\Delta\gamma}\right)l^2 + (l - l_0)hl_0}{6A_0 + h(2l_0 + l)}$$

$$p = \frac{2\Delta\gamma A}{l}\left(2 - \frac{3u}{l}\right) \text{(limiting intensity)} \qquad p = \gamma\left(\frac{2\Delta A}{l} - cH\right)\left(2 - \frac{3u}{l}\right)$$

$$q = \frac{2\Delta\gamma A}{l}\left(2 - \frac{3y}{l}\right) \text{(limiting intensity)}$$

If T enter the following formulæ, H above becomes H_1. (See Figs. 11 and 12.)

The first column of formulæ just given would apply, with the condition of upward pressure on the base due to hydrostatic head, if a proper value of u corresponding, be

taken, that is if the excursion of the force $A\Delta\gamma$, resulting from the effect of all other forces on $A\Delta\gamma$ be considered, rather than the effect of all the other forces on the resultant vertical force.

It will be found expeditious to design a section, where ice pressure at the level of full reservoir is to be considered in connection with the water surface at some higher flood level, first by series of formulæ containing T (cf. Series B and D) and then to investigate successive bases, or joints, thus obtained (beginning, for a high masonry dam, usually at a base, or joint, about 100 feet from the top of the dam) with series of formulæ lacking T, or the ice pressure condition (cf. Series A and C). A base will ultimately be obtained by these supplementary "Flood level" calculations greater than the base at its same elevation as previously determined by the "Ice Pressure" design.

Continuing with the design by means of the "Flood level" formulæ to the bottom of maximum height required will determine the minimum cross-section area to meet the conditions both of "Flood" and of "Ice." It should be remarked in this connection that when a reservoir level is rising due to flood conditions prevailing, it is evident that ice formation cannot develop, or, in other words, the two conditions cannot be coexistent, hence the difference in designation of hydrostatic heads corresponding. (See Figs. 11 and 12.)

Series A.

Conditions: Overturning moment due to *horizontal static water pressure on back of dam only.*

Stage I.

$$H = \sqrt[3]{\Delta L^2(H+a)}.$$

Stage II.

$$l^2 + \left(\frac{4A_0}{h} + l_0\right)l = \frac{1}{h}\left(\frac{H^3}{\Delta} + 6A_0y_0\right) + l_0^2.$$

Stage III.

$$l^2 + \left(\frac{2A_0}{h} + l_0\right)l = \frac{H^3}{\Delta h}.$$

Stage IV.

$$l^2 = \frac{\gamma H^3}{p}.$$

Stage V.

$$\left(\frac{p+q}{h\Delta\gamma} - 1\right)l^2 - \left(\frac{2A_0}{h} + l_0\right)l = \frac{H^3}{\Delta h}.$$

Series B.

Conditions: Overturning moment due to:

(*a*) *Horizontal static water pressure on back* and

(*b*) *Ice pressure applied at distance* (a_1) *from top.*

Stage I.

$$H_1 = \sqrt[3]{\Delta(H_1 + a_1)L^2 - 6TH_1}.$$

Stage II.

$$l^2+\left(\frac{4A_0}{h}+l_0\right)l=\frac{1}{h}\left(\frac{H_1{}^3+6TH_1}{\Delta}+6A_0y_0\right)+l_0{}^2.$$

Stage III.

$$l^2+\left(\frac{2A_0}{h}+l_0\right)l=\frac{1}{\Delta h}(H_1{}^3+6TH_1).$$

Stage IV.

$$l^2=(H_1{}^3+6TH_1)\frac{\gamma}{p}.$$

Stage V.

$$\left(\frac{p+q}{h\Delta\gamma}-1\right)l^2-\left(\frac{2A_0}{h}+l_0\right)l=\frac{1}{\Delta h}(H_1{}^3+6TH_1).$$

Series C.

Conditions: Overturning moment due to:

(*a*) *Horizontal static water pressure on back.*

(*b*) *Upward water pressure on base.* Pressure intensity decreasing uniformly from $cH\gamma$ at heel to zero intensity at toe.

Stage I.

$$H=\sqrt[3]{L^2[\Delta(H+a)-cH]}.$$

Stage II.

$$\left(1-\frac{cH}{\Delta h}\right)l^2+\left(\frac{4A_0}{h}+l_0\right)l=\frac{1}{h}\left(\frac{H^3}{\Delta}+6A_0y_0\right)+l_0{}^2.$$

Stage III.

$$\left(1-\frac{cH}{\Delta h}\right)l^2+\left(\frac{2A_0}{h}+l_0\right)l=\frac{H^3}{\Delta h}.$$

Stage IV.

(*a*) $$\left(1-\frac{cH}{\Delta h}\right)l^3+\left(\frac{2A_0}{h}+l_0\right)l^2-\frac{\gamma H^3}{p}\left(1-\frac{cH}{\Delta h}\right)l$$
$$=\frac{\gamma H^3}{p}\left(\frac{2A_0}{h}+l_0\right), \text{ which reduces to}$$

(*b*) $$l^2=\frac{\gamma H^3}{p}.$$

Stage V.

(*a*) $$\left(1-\frac{cH}{\Delta h}\right)\left(1-\frac{p+q}{h\Delta\gamma}\right)l^3+\left(\frac{2A_0}{h}+l_0\right)\left(2-\frac{p+q}{h\Delta\gamma}-\frac{cH}{h\Delta}\right)l^2$$
$$+\left[\frac{2A_0}{h}\left(\frac{2A_0}{h}+2l_0\right)+\frac{H^3}{h\Delta}\left(1-\frac{cH}{h\Delta}\right)+l_0^2\right]l$$
$$=-\frac{H^3}{h\Delta}\left(\frac{2A_0}{h}+l_0\right), \text{ which reduces to}$$

(*b*) $$\left(\frac{p+q}{h\Delta\gamma}-1\right)l^2-\left(\frac{2A_0}{h}+l_0\right)l=\frac{H^3}{\Delta h}.$$

Series D.

Conditions: Overturning moment due to:

(*a*) *Horizontal static water pressure on back* (head $=H_1$).

(*b*) *Ice pressure applied at distance* (a_1) *from top.*

(*c*) *Upward water pressure on base.* Pressure decreasing uniformly from $cH_1\gamma$ at heel to zero intensity at toe.

Stage I.

$$H_1=\sqrt[3]{L^2[(H_1+a_1)\Delta-cH_1]-6TH_1}.$$

Stage II.

$$\left(1-\frac{cH_1}{\Delta h}\right)l^2+\left(\frac{4A_0}{h}+l_0\right)l=\frac{1}{h}\left(\frac{H_1^3+6TH_1}{\Delta}+6A_0y_0\right)+l_0^2.$$

Stage III.

$$\left(1-\frac{cH_1}{\Delta h}\right)l^2+\left(\frac{2A_0}{h}+l_0\right)l=\frac{1}{\Delta h}(H_1{}^3+6TH_1).$$

Stage IV.

$$(a)\quad \left(1-\frac{cH_1}{\Delta h}\right)l^3+\left(\frac{2A_0}{h}+l_0\right)l^2-\frac{\gamma(H_1{}^3+6TH_1)}{p}\left(1-\frac{cH_1}{\Delta h}\right)l$$

$$=\frac{\gamma(H_1{}^3+6TH_1)}{p}\left(\frac{2A_0}{h}+l_0\right), \text{ which reduces to}$$

$$(b)\quad l^2=\frac{\gamma(H_1{}^3+6TH_1)}{p}.$$

Stage V.

$$(a)\quad \left(1-\frac{cH_1}{\Delta h}\right)\left(1-\frac{p+q}{h\Delta\gamma}\right)l^3+\left(\frac{2A_0}{h}+l_0\right)\left(2-\frac{p+q}{h\Delta\gamma}-\frac{cH_1}{h\Delta}\right)l^2$$

$$+\left[\frac{2A_0}{h}\left(\frac{2A_0}{h}+2l_0\right)+\frac{(H_1{}^3+6TH_1)}{h\Delta}\left(1-\frac{cH_1}{h\Delta}\right)+l_0{}^2\right]l$$

$$=-\frac{(H_1{}^3+6TH_1)}{\Delta h}\left(\frac{2A_0}{h}+l_0\right), \text{ which reduces to}$$

$$(b)\quad \left(\frac{p+q}{h\Delta\gamma}-1\right)l^2-\left(\frac{2A_0}{h}+l_0\right)l=\frac{H_1{}^3+6TH_1}{\Delta h}.$$

In the preceding series of equations it will be observed that the final expressions for l in stages IV and V are very similar, and that the quantity c in equations (a) of these stages disappears in equations (b). Equations (b) of course, are to be used for purposes of calculation of cross-sections.

SERIES E.

Conditions: (See Fig. 13.) Ice pressure neglected in Fig. 13. Overturning moment due to:

(*a*) *Horizontal static water pressure on back* (head $= h_1$).

(*b*) *Ice pressure applied at distance* (a_1) *from top.*

(*c*) *Upward water pressure on base;* pressure intensity decreasing uniformly from $c(h_1+h_2)\gamma$ at heel to zero intensity at toe.

(*d*) *Mud* (*liquid*) *pressure on back* (head h_2), commencing at distance h_2 above joint in question. Weight of mud $= \gamma'$. As before, if T be equated to zero, a_1 becomes equal to a, in the formulæ.

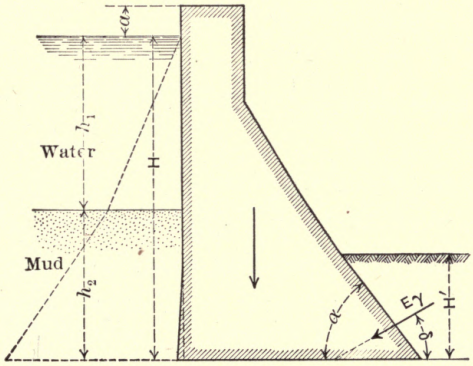


FIG. 13.

Stage I. (h_1, of known value, h_2 to be determined.)

$$\frac{h_2^3\gamma'}{\gamma}+(h_1+h_2)[3h_1h_2+6T+L^2(c-\Delta)]=L^2a_1\Delta-h_1^3.$$

Stage II.

$$\left[1-\frac{c(h_1+h_2)}{\Delta h}\right]l^2+\left(\frac{4A_0}{h}+l_0\right)l$$
$$=\frac{1}{\Delta h}\left[(h_1+h_2)(3h_1h_2+6T)+h_1^3+\frac{h_2^3\gamma'}{\gamma}+6A_0y_0\Delta\right]+l_0^2.$$

For trapezoidal section at top, make $A_0 = 0$ and $y_0 = 0$ and $l_0 = L$ in Stage II. This applies generally.

Stage III.

$$\left[1 - \frac{c(h_1 + h_2)}{\Delta h}\right] l^2 + \left(\frac{2A_0}{h} + l_0\right) l$$
$$= \frac{1}{\Delta h}\left[(h_1 + h_2)(3h_1h_2 + 6T) + h_1^3 + \frac{h_2^3\gamma'}{\gamma}\right].$$

Stage IV.

$$l^2 = \frac{\gamma}{p}\left[(h_1 + h_2)(3h_1h_2 + 6T) + h_1^3 + \frac{h_2^3\gamma'}{\gamma}\right].$$

Stage V.

$$\left(\frac{p+q}{h\Delta\gamma} - 1\right) l^2 - \left(\frac{2A_0}{h} + l_0\right) l$$
$$= \frac{1}{\Delta h}\left[(h_1 + h_2)(3h_1h_2 + 6T) + h_1^3 + \frac{h_2^3\gamma'}{\gamma}\right].$$

From a study of the formulæ thus far developed it will be observed that by reducing certain conditions to zero, with their corresponding quantities, the main equations of a given series reduce to those of a simpler series.

For instance—

In Series B make T (for ice pressure condition of loading) equal to zero and $H_1 = H$ and $a_1 = a$ and the main equations of that series reduce to Series A equations.

In Series C, by making c (for upward water pressure condition) equal to zero in main equations of Stages I, II, and III and also in equations (a) of Stages IV and V, the equations of Series C reduce to those of Series A.

Likewise, by making the proper eliminations and substitutions, Series E will reduce to Series D, C, B or A.

Series F.

This series consists of general formulæ for a number of imposed conditions of loading. For any given case, the terms or factors expressing those conditions not appertaining must be eliminated by equating them to zero. (See Fig. 14.)

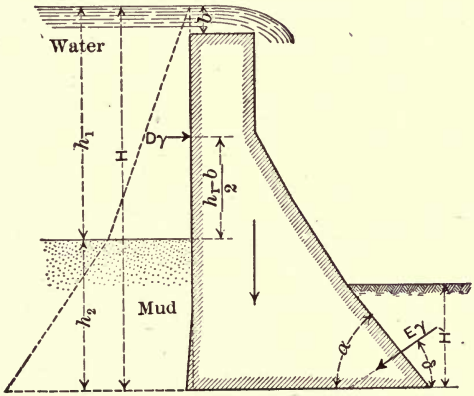


Fig. 14.

Conditions for General Formulæ.

Overturning moment due to:

(*a*) *Horizontal static water pressure on back* (head $= h_1$).

(*b*) *Upward water pressure on base;* pressure intensity decreasing uniformly from $cH\gamma$ or $c(h_1+h_2)\gamma$, at heel to zero intensity at toe.

(*c*) *Mud (liquid) pressure on back* (head h_2) as before.

(*d*) *Dynamic pressure of water*, $D\gamma$.

(*e*) *Water flowing over top of dam*, weight of water, of depth b, on top of dam being neglected.

For condition of water *not* overtopping dam, $b=0$ and $D=0$.*

For condition of no dynamic pressure, $D=0$.

For condition of no upward water pressure, $c=0$.

For condition of no mud (i.e., mud being replaced by water) make $h_2=0$, $h_1=H$.

Stage I.

Rectangular cross-section at top or rectangular dam, $l=l_0=L$.

This may fall under either of two cases, viz.—

Case (1)

Condition: $h_1=H$; $h_2=0$.

$$H^3+H[+3D-3b^2+L^2(c-\Delta)]=b(3D-2b^2-L^2\Delta).$$

Case (2)

Condition: h_1 of known value; h_2 to be determined.

$$\frac{h_2^3\gamma'}{\gamma}+3Dh_2+(h_1+h_2)[3h_1h_2+3D-3b^2+L^2(c-\Delta)]$$
$$=b(3D-2b^2-L^2\Delta)-h_1^3.$$

As in the preceding series, the value of H or h_2, of Stage I may be determined by several successive trial substitutions.

Stage II.

(*a*) Trapezoidal cross-section at top of dam or trape-

* For water surface below top of dam, Series E, containing the distance "a" must be used.

zoidal dam (spillway) front face battered. ($A_0 = 0$, $l_0 = L$ and $y_0 = 0$.) Note: For a triangular dam $l_0 = 0$, also.

$$\left[1 - \frac{c(h_1+h_2)}{\Delta h}\right] l^2 + Ll = \frac{1}{\Delta h}\left[(h_1+h_2)(3h_1h_2 - 3b^2) + h_1^3 + \frac{h_2^3\gamma'}{\gamma} + 2b^3 + 3D(h_1 + 2h_2 - b)\right] + L^2.$$

(*b*) Trapezoidal section continued (front face battered).

$$\left[1 - \frac{c(h_1+h_2)}{\Delta h}\right] l^2 + \left(\frac{4A_0}{h} + l_0\right) l = \frac{1}{\Delta h}\left[(h_1+h_2)(3h_1h_2 - 3b^2) + h_1^3 + \frac{h_2^3\gamma'}{\gamma} + 2b^3 + 3D(h_1 + 2h_2 - b) + 6A_0y_0\Delta\right] + l_0^2.$$

Stage III.—Both faces battered.

$$\left[1 - \frac{c(h_1+h_2)}{\Delta h}\right] l^2 + \left(\frac{2A_0}{h} + l_0\right) l = \frac{1}{\Delta h}\left[(h_1+h_2)(3h_1h_2 - 3b^2) + h_1^3 + \frac{h_2^3\gamma'}{\gamma} + 2b^3 + 3D(h_1 + 2h_2 - b)\right].$$

Stage IV.—Limiting intensity of pressure, p, introduced.

$$l^2 = \frac{\gamma}{p}\left[(h_1+h_2)(3h_1h_2 - 3b^2) + h_1^3 + \frac{h_2^3\gamma'}{\gamma} + 2b^3 + 3D(h_1 + 2h_2 - b)\right].$$

Stage V.—Limiting intensities, p and q.

$$\left(\frac{p+q}{h\Delta\gamma} - 1\right) l^2 - \left(\frac{2A_0}{h} + l_0\right) l = \frac{1}{\Delta h}\left[(h_1+h_2)(3h_1h_2 - 3b^2) + h_1^3 + \frac{h_2^3\gamma'}{\gamma} + 2b^3 + 3D(h_1 + 2h_2 - b)\right].$$

The increased number of overturning loads then, tend to render the right-hand members of the various equations more involved; though after a little practice one may easily carry through a design with surprising rapidity. The slide rule may be used to great advantage and it is suggested that the results be tabulated as determined, in some such form as the following:

TABLE OF RESULTS.

$c=$, $T=$, etc.

Joint No.	H	H^3	h	$H+a$	A_0	A	l_0	l_0^2	l	y_0	y	u	t	p	q	A_0y_0	Etc.
1																	
2																	
3																	
4																	
Etc.																	

CHAPTER IV

FORMULÆ FOR INVESTIGATION

THE effect upon the calculation of a cross-section, of backfill on the down-stream face, could, of course, be cared for by introducing that condition into the preceding series of equations; but as this effect as computed, would be, in any case, largely dependent upon assumptions which may vary widely and as the placing of backfill is generally a later consideration with respect to construction, the propriety of such introduction at that stage of design is questionable.

In the following formulæ for *investigation* therefore, the general conditions of an earth thrust acting at the down-stream face and of a vertical component of thrust of material on the up-stream, inclined face of the dam, are introduced. Moments of forces are taken about a point in the joint distant y from the up-stream edge of the joint in derivations for u.

By any of these formulæ the position of the line of resistance for any given cross-section and respective conditions may be determined with regard to any horizontal joint and its down-stream edge; the value of u being the quantity to be sought.

Any condition may be disregarded by equating its term to zero.

The first expression below contains all of the conditions heretofore considered with the additional ones just stated; and from it follow the succeeding expressions for u. It should be remembered that the term T cannot be coexistent in any expression for stability with b and therefore with D. Nevertheless all of these terms are written with the understanding that the proper eliminations be always made. Three general group equations will be written.

Formulæ for Investigation.

First, Conditions of retained mud, water, overtopping, etc. (see Fig. 14).

$$u = l - y - \frac{\left\{ \begin{array}{l} (h_1+h_2)[3h_1h_2+6T-3b^2+cl(3y-l)]+h_1^3 \\ +\dfrac{h_2^3\gamma'}{\gamma}+2b^3+3D(h_1+2h_2-b)-6W_v(y-s) \\ +6E\left[(l-y)\sin\delta-\dfrac{H'}{3}\dfrac{\sin(\delta+\alpha)}{\sin\alpha}\right] \end{array} \right\}}{6(W_v+E\sin\delta+A\Delta)-3c(h_1+h_2)l}.$$

Whence, for conditions of *retained mud, water, etc.*, but *no overtopping*, by making $b=0$ and $D=0$, there follows (see Fig. 13):

$$u = l - y - \frac{\left\{ \begin{array}{l} (h_1+h_2)[3h_1h_2+6T+cl(3y-l)]+h_1^3+h_2^3\dfrac{\gamma'}{\gamma} \\ -6W_v(y-s)+6E\left[(l-y)\sin\delta-\dfrac{H'}{3}\dfrac{\sin(\delta+\alpha)}{\sin\alpha}\right] \end{array} \right\}}{6(W_v+E\sin\delta+A\Delta)-3c(h_1+h_2)l}.$$

From this last expression for u, for conditions of *retained water, etc.*, but *neither mud nor overtopping*, by making $h_1=H_1$; $h_2=0$, there is obtained:

$$u = l - y - \frac{\left\{6T + H_1^2 + cl(3y - l) - \frac{6W_v}{H_1}(y - s) + \frac{6E}{H_1}\left[(l - y)\sin\delta - \frac{H'}{3}\frac{\sin(\delta + \alpha)}{\sin\alpha}\right]\right\}}{6\left(\frac{W_v + E\sin\delta + A\Delta}{H_1}\right) - 3cl}.$$

As in the equations for design, when $T = 0$, $H_1 = H$. (See Figs. 11 and 12.) If H' is of such depth that the down-stream batter of the cross-section varies considerably, an approximate solution is possible by assuming some average batter for the lower portion. The expression for earth thrust is general, as is evidenced. After u is determined for each joint, the intensities of maxima pressures can be determined for the given cross-section, the general expression for p, corresponding to above expressions for u, being:

$$p = \frac{2\gamma}{l}\left[W_v + E\sin\delta + A\Delta - \frac{c(h_1 + h_2)}{2}l\right]\left(2 - \frac{3u}{l}\right).$$

In connection with the computation for the value of y in an investigation, as indicated above, it is necessary to obtain the position of the centroid of a trapezoid with respect to the back, or up-stream edge, of the joint in question. The following expression for x, in connection with Fig. 15, may prove convenient:

$$x = \frac{(l^2 + l\,l_0 + l_0^2) + t(l + 2l_0)}{3(l + l_0)}.$$

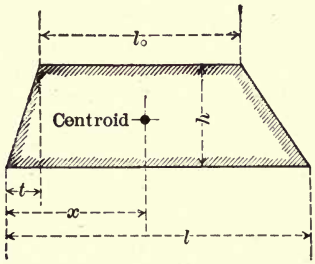


FIG. 15.

It is desirable to consider tension as active in the joint, and if p'_t is the intensity, in tons per square foot,

at the down-stream end of the joint, and p''_i is the intensity at the up-stream edge of the joint, p, above, which, as written, is in pounds per square foot, will take the form:

$$p_i' = \frac{1}{16l}\left[W_v + E \sin \delta + A\Delta - \frac{c(h_1+h_2)l}{2}\right]\left[2 - \frac{3u}{l}\right],$$

and

$$p_i'' = \frac{1}{16l}\left[W_v + E \sin \delta + A\Delta - \frac{c(h_1+h_2)l}{2}\right]\left[\frac{3u}{l} - 1\right].$$

In the case where there is liquid mud only on the back, W_v becomes equal to $\frac{A'\gamma'}{\gamma}$, where A' is the area of the superimposed mud.

In the use of the foregoing formulæ, it may be desirable to take account of some such variation of the extent of the uplift intensity on the base or joints, as was indicated in Chapter I, p. 14, in the distinction to be drawn between uplift conditions in the foundations and higher up in the dam.

For the foundation, a trapezoidal distribution was suggested and for the body of the masonry, drained by wells, a triangular distribution of intensities, but of shortened extent down-stream, was proposed.

In the former case, the total pressure would be increased, but its lever arm would tend to be diminished. In the latter case, the pressure would be diminished, but the lever arm increased. The effect on a cross-section design, or on a line of pressure for a given cross-section, would have to be worked out for any particular case. For such a structure as the Kensico Dam, the foregoing changes from the ordinary triangular assumption, while modifying the numerical results as to "factors" against overturning

and as to resulting pressure intensities on the various joints and the foundation, did not modify the final cross-section.

An approximation, however, may be satisfactorily obtained by varying the value of c as one works down the cross-section from the top. The amount of the requisite variation may be ascertained by comparing the overturning moment of the uplift, acting as usually assumed, with the value of the overturning moment as desired to be assumed in the given case. This may be done for three different points down the dam, and the ratio found by this comparison. A curve may then be plotted in terms of these ratios, and the distances from the top, all figured from a cross-section assumed as nearly like the dam under consideration as can be anticipated. A curve through the three points will yield the relative values of the variable c's to be used in the formulæ, each c for its own elevation. Or else formulæ such as these here given may be derived for the given assumption and used directly.

In the studies for design, referred to before, the analytic work should be checked throughout by the graphic method wherever possible. This should always be done both in designing and investigating cross-sections.

It should be stated here that, after a cross-section has been fixed upon for a given dam and the faces drawn to chosen batters and curves, the entire cross-section should be investigated as just outlined so as to give the actual values for this final cross-section.

Again, in comparing different cross-sections, especially of different dams, by superimposing, their *water lines* should be made to coincide and not their tops for a fair comparison.

CHAPTER V

THE DESIGN OF A HIGH MASONRY DAM

To illustrate the method of applying the preceding formulæ to the determination of the theoretical cross-section of a high masonry dam, an actual problem will be presented. For this purpose the Olive Bridge Dam has been selected, not only because it is representative of the type for which the formulæ were developed, but because the structure has been recently put into service, and is sufficiently well known to be of more than passing interest.

It may not be inappropriate, before proceeding to the computations, to refer to certain of the structure's more important features, especially as some were entirely new, and to give a brief description of it.

The Olive Bridge Dam is the principal structure of a number of dams and dikes which serve to impound the waters of the Ashokan Reservoir. The latter is located about 14 miles west of the Hudson River at Kingston, N. Y., has an available storage capacity of 128 billion gallons, derived from the Esopus watershed, with an area of 255 square miles, and delivers the stored water to the Catskill Aqueduct, whose capacity is 500,000,000 gallons per day, to be conducted to the City of New York, about 100 miles away, and on the opposite side of the Hudson River.

The reservoir is formed by the Olive Bridge Dam, placed across Esopus Creek, and by the West, Middle, East and Hurley dikes, located in the smaller gaps between the hills which create the natural basin.

The dividing dike with its weir separates the reservoir into two portions, while the waste weir, nearly 1000 feet long, at the eastern end of the large dikes, provides the means of discharging surplus floods safely. All of the weirs are masonry structures.

The length of the reservoir is 12 miles, with a maximum width of 3 miles, while the total length of the dam and the dikes is 5½ miles.

Work was started in the latter part of 1907, and on Sept. 9, 1913, the storage of water in the west basin began. By Oct. 2 in the same year, the first Esopus water could have been delivered by gravity into the Catskill Aqueduct, the water surface of the west basin having reached elevation 495, or a depth of about 95 feet behind the dam, the equivalent of 2100 million gallons impounded, but unavailable.

Three types of dam were considered in the studies.

(1) An earth dam, to be constructed by sluicing or some other method.

(2) A composite dam, or one consisting of a masonry core, covered by an earth embankment, the masonry portion being of small section across Esopus gorge and rising to within 50 feet of the water surface for full reservoir, the earth embankment making up the remainder.

(3) A masonry dam of the gravity type, extending across the gorge and flanked by earth wings at each side of the valley.

The investigations were so far advanced at the end of 1906 that a decision in favor of the third type was reached early in 1907, after 268 study drawings had been made.

The masonry dam is founded on solid ledge rock. A cut-off trench extends about 40 feet below the stream bed, and grout holes go 20 feet deeper. The main structure is of cyclopean masonry, with concrete face blocks, while the wings, built of acceptable earth found near the site, contain concrete corewalls extending to solid ledge or into very compact impervious earth.

The masonry portion of the main dam, which rises to an elevation of 210 feet above the stream bed, is 1000 feet long, but the total length of the structure including the earth wings is 4650 feet.

To prevent temperature cracks in the masonry section, it was decided to divide the dam transversely by means of vertical "expansion joints," the distance between which would be well within those distances at which cracks had heretofore been observed in other structures. These intervals varied from 84 to 91 feet.

The expansion joints (more properly termed contraction joints) are formed by building vertical faces of concrete blocks, shaped as a tongue-and-grooved joint, normal to the axis of the dam, and thus preventing a continuous opening through the structure, should the adjacent sections contract. Vertical inspection wells at each expansion joint afford the opportunity of studying the conditions at these sections.

There are also two longitudinal inspection galleries built within the dam, one near the top and entered by manholes from the surface, and one near the lower portion

of the dam, both of which are connected with the vertical inspection wells.

The inspection wells afford the opportunity, should it be deemed desirable, of placing a copper strip across each expansion joint, to reduce the quantity of water that may be expected to pass through them.

The lower longitudinal gallery opens near the center of the dam into a transverse gallery, leading to a measuring weir chamber and drain, at the downstream side, where the entire leakage may be gaged and discharged.

Between the vertical inspection wells, drainage wells 16 inches in diameter, and about 12 feet apart, slightly inclined downstream from the top, are provided, between the upper and lower longitudinal galleries, to intercept seepage into the masonry and to prevent any water from reaching, and consequently disfiguring, the downstream face of the structure.

These wells were constructed by laying up large, hollow, porous concrete blocks.

Small quantities of water which may enter the body of the dam, either through the expansion joints, into the inspection wells, or through the capillary spaces in the masonry, will be conducted by means of the wells, galleries and drains to the gorge below the dam.

It is proposed later to fill the vertical inspection wells with material that will effectively stop all flow.

The following calculations indicate in detail the method pursued in the determination of the theoretical cross-section of the Olive Bridge Dam.

The conditions that governed in the design were as follows:

1.	Elevation of top of dam above datum	610 ft.
2.	Elevation of free water surface, reservoir full	590 ft.
3.	Elevation of free water surface, reservoir in max. flood	596 ft.
4.	Elevation of free water surface, reservoir in max. flood and wind blowing at max. velocity of 40 miles per hour, with waves piled up	600 ft.
5.	Elevation of bottom of max. section; rock excavation at this point approximately 10 feet deep	390 ft.
6.	Ice pressure, assumed the same as that used in the design of Wachusett Dam, applied at elevation 590.	23.5 tons per sq.ft.
7.	Upward pressure due to hydrostatic head assumed to be of uniformly varying intensity, and varying from a max. at the heel of the joint to zero at the toe, distributed over $\frac{2}{3}$ the area of the joint.	
8.	At the maximum section the earth is refilled to the top of the gorge equivalent to elevation 450 or 500.	
9.	Other conditions so liberal that section will be reasonably safe against earthquake and dynamite.	
10.	Maximum allowable unit pressure	20 tons per sq.ft.
11.	Top width	23 ft.
12.	Resultant line of pressure, reservoir full and empty, shall lie within the middle third at each joint.	
13.	Δ, or ratio of unit weight of masonry to unit weight of water	$\frac{7}{3}$
14.	γ, weight of a cubic foot of water	62.5 lbs.
15.	c, ratio of upward thrust intensity due to hydrostatic head, assumed to act at heel of joint	$\frac{2}{3}$

Flood conditions, Series C, and ice conditions, Series D, will be imposed, and the design prosecuted with respect to each simultaneously.

From the above it follows that

$$a = 10 \text{ feet (flood conditions)},$$

$$a_1 = 20 \text{ feet (ice conditions)},$$

$$6T = 4512,$$

$$\frac{c}{\Delta} = \frac{2}{7}.$$

Joint No. 1, Flood Conditions

Generally speaking it may be assumed that the top of a masonry dam will be about $\frac{1}{10}$ of the height above the

water, but in the case under consideration, a superelevation of only 20 feet was employed for full reservoir, and 10 feet for flood conditions. While the choice in this respect is purely arbitrary, the above ratio is the one usually prescribed if there are no other governing conditions.

Since the length of a joint depends upon its depth below the water surface, it is evident that at the surface this dimension should be zero. For various reasons, however, such as the desirability of a foot-walk or a driveway on the crest, a top width is chosen which will satisfy the demands.

As 23 feet had been decided upon as the top width, for a considerable distance below the water surface the rectangular section will more than satisfy the only condition for stability that applies in this portion of the dam, namely, that the resultant of all the external forces for reservoir full and reservoir empty shall be within the middle third of the cross-section. It is evident that for reservoir empty, the resultant passes through the center of the joint. It becomes necessary, however, for reservoir full, to determine the depth H at which this resultant first emerges from the middle third of the rectangular section.

For flood conditions we will use the equation under Stage I, Series C, which is,

$$H = \sqrt[3]{L^2[\Delta(H+a) - cH]}.$$

Here $a = 10$ feet, $\Delta = \frac{7}{3}$, $L = 23$ feet; and $c = \frac{2}{3}$.

This equation may be solved by successive substitutions for H, until such a value of H is found that equality results. In the present instance it is found that $H = 35.1$ feet satisfies the equation, and hence the rectangular cross-

section of the dam may be carried down to a depth of 45.1 feet below the top, since a, the superelevation under flood conditions, is equal to 10 feet.

Joint No. 1*a*, Ice Conditions

In a similar manner for ice conditions, we must use the following formula, under Stage I, Series D, to determine the depth, H_1, at which the resultant first emerges from the middle third of the rectangular section.

$$H_1 = \sqrt[3]{L^2[(H_1+a_1)\Delta - cH_1] - 6TH_1}.$$

Here $a_1 = 20$ feet, $\Delta = \frac{7}{3}$, $L = 23$ feet, $c = \frac{2}{3}$ and $6T = 4512$.

By successive substitutions for H_1, it is found that a value of 6.7 feet will satisfy the equation and consequently the rectangular section in this case can be carried down only 6.7 feet below full reservoir, or, since the superelevation a_1 is 20 feet, to a depth of only 26.7 feet below the top, before it will be necessary to modify the section.

A comparison of the two values established, $H_1 + a_1 = 26.7$ feet, and $H + a = 45.1$ feet, together with an examination of the profile, shows the very marked effect the assumption of ice pressure has upon increasing the cross-section in the upper levels of the dam.

The solution for either H or H_1 may be expedited by the use of the graphic method. Thus, assume at least three values for H, say 30 feet, 40 feet, and 50 feet in the present case, substitute successively in the right-hand member of the above equation and solve. Plot these resulting values as abscissæ and the assumed values for H corresponding as ordinates. A smooth curve drawn through the points thus obtained will give a point where

ordinate and abscissæ are equal, and this will locate the desired value.

At no point in this portion of the dam does the length of a horizontal joint change, but below a depth of 26.7 feet from the top, this dimension will have to be increased in order to comply with the requirement for stability, which prescribes that the resultant of all external forces shall lie within the middle third. This is accomplished by giving a batter to the downstream face of the dam, while the upstream face remains vertical.

This stage of the design extends from the lower limits of the rectangular section to that elevation where it first becomes necessary to batter the back, and the formulæ to be used are those found under Stage II, Series C, for flood, and Series D for ice conditions.

Joint No. 2, Flood Conditions

The investigation for the purpose of determining the length l of joint No. 2 involves the use of an equation in which u shall have a value of $\frac{l}{3}$, since the resultant of the external forces for the reservoir full reached the limit of the middle third at the downstream side at joint No. 1, and since, also, it may not pass outside that limit. This is expressed for flood conditions by the following equation from Stage II, Series C:

$$\left(1-\frac{cH}{\Delta h}\right)l^2+\left(\frac{4A_0}{h}+l_0\right)l=\frac{1}{h}\left(\frac{H^3}{\Delta}+6A_0y_0\right)+l_0^2.$$

The value of 4.9 feet will be given to h, to bring the depth of joint No. 2 to elevation 560, or 40 feet below the

flood level, and 50 feet below the top of the dam, merely because this is an easier figure to work with and because the depth of each successive joint is taken as 10 feet, or some multiple of 10 feet, below the next above.

In the above expression the factors take the following values:

$$A_0 = 1035 \text{ square feet}; \quad H = 40 \text{ feet}; \quad h = 4.9 \text{ feet};$$
$$l_0 = 23 \text{ feet}; \quad y_0 = 11.5 \text{ feet}; \quad \text{and} \quad \Delta = \tfrac{7}{8}.$$

Substituting these values in the above, completing the square, and solving for l, we obtain:

$$l = 25 \text{ feet}.$$

As it is necessary to use the factor A at each successive joint, it should be determined at this point

$$A = A_0 + \frac{l_0 + l}{2} h.$$

All the quantities in the above equation are known, and A is found to equal 1152.5 square feet.

It will be noted that the equation employed to determine the value of l involves the factor y_0, which represents the distance from the up-stream face to the point of application of the resultant pressure for reservoir empty, on the joint next above the one being investigated. It therefore becomes necessary after each application of the formula to use the supplementary equation from which the value of y is found, not only for the above purpose of locating the point of application of the resultant, reservoir empty, but because this y becomes in turn y_0 for the joint next below.

Furthermore, as w has already been given the value of $\frac{l}{3}$, the only condition for stability in this section of the dam involving Stage II is that y shall be equal to or greater than $\frac{l}{3}$.

$$y = \frac{A_0 y_0 + \frac{h}{6}(l^2 + l l_0 + l_0^2)}{A_0 + \left(\frac{l + l_0}{2}\right)h}.$$

As all of these factors are already known, it merely requires that they be substituted in the above, which results in

$$y = 11.6 \text{ feet.}$$

This indicates that the pressure line P', for reservoir empty, lies well within the middle third of Joint No. 2.

As the limiting depth of the rectangular section under ice conditions was found to be 26.7 feet below the top of the dam, while for flood conditions it was located 45.1 feet below the top, it will be necessary to investigate the joint at the latter point and called joint No. 1, under the ice conditions, to see what effect the ice will have upon its length. After the l has been determined for this joint, the length of the joint under ice conditions at a depth of 50 feet below the top, called joint No. 2, will be found, so that both joint No. 1 and joint No. 2 will have values of l under flood conditions and under ice conditions, and thereafter the procedure will be to investigate the length of each successive joint, first, for flood and then for ice.

The following formula from Stage II, Series D, will be employed:

JOINT NO. 1, ICE CONDITIONS

$$\left(1-\frac{cH_1}{\Delta h}\right)l^2+\left(\frac{4A_0}{h}+l_0\right)l=\frac{1}{h}\left(\frac{H_1^3+6TH_1}{\Delta}+6A_0y_0\right)+l_0^2.$$

In this expression the following factors are known: $c=\frac{2}{3}$, $\Delta=\frac{7}{3}$, $H_1=25.1$ feet, $h=18.4$ feet, $A_0=615$ square feet, $6T=4512$, $y_0=11.5$ feet, $l_0=23$ feet; whence, by substitution, completing the square, and solving for l, we obtain the value

$$l=33.0 \text{ feet.}$$

To determine A, we have

$$A=A_0+\frac{l_0+l}{2}h=1130 \text{ square feet.}$$

The preceding determination of the value of l, shows that at a depth of 45.1 feet below the top of the dam the length of the same joint should be 23 feet under flood conditions, and 33 feet when ice is assumed to act.

We must now determine y, since its value must be used in the joint next below where it takes the symbol y_0, and the following formula is employed, in which all the quantities are known:

$$y=\frac{A_0y_0+(l^2+ll_0+l_0^2)\frac{h}{6}}{A_0+\left(\frac{l+l_0}{2}\right)h}=12.7 \text{ feet.}$$

It should be noted here that the denominator in the above is equal to A.

Similarly for

Joint No. 2, Ice Conditions

Using the same formulæ for l as in joint No. 1 with new values to some of the factors, thus:

$H_1 = 30$ feet, $h = 4.9$ feet, $l_0 = 33$ feet, $A_0 = 1130$ square feet, $y_0 = 12.7$ feet.

We find that

$$l = 35.5 \text{ feet.}$$

Computing the value of A

$$A = A_0 + \frac{l_0 + l}{2} h = 1298 \text{ square feet,}$$

and solving for y with the factors from above and using $A = 1298$ square feet for the denominator, we obtain

$$y = 13.3 \text{ feet.}$$

Joint No. 3, Flood Conditions

The same conditions apply to this joint as for Joint No. 2, so that the same equations will have to be used. In the equation for l, Series C, the factors that change have the following values:

$$A_0 = 1153 \text{ square feet, } H = 50 \text{ feet, } h = 10 \text{ feet,}$$
$$l_0 = 25 \text{ feet, and } y_0 = 11.6 \text{ feet,}$$

and by substituting them in the equation, completing the square, and solving for l, we will obtain

$$l = 29.5 \text{ feet.}$$

Using the equation for A we find its value to be

$$A = 1426 \text{ square feet.}$$

Similarly, using the equation for y, with the new values for the variables as indicated above we have,

$$y = 12 \text{ feet.}$$

Joint No. 3, Ice Conditions

Using the same equation here that was used for Joint No. 2 (Stage II Series D) under ice conditions with the factors taking the following values: $A_0 = 1298$ square feet, $H_1 = 40$ feet, $h = 10$ feet, $l_0 = 35.5$ feet, and $y_0 = 13.3$ feet, we obtain the following value for l:

$$l = 40 \text{ feet},$$

while the use of the same factors in the determination of A gives

$$A = 1676 \text{ square feet},$$

and y is found to have the value of

$$y = 14.5 \text{ square feet}.$$

At this point it would be appropriate to determine whether the maximum allowable pressure per square foot had been reached, and as the ice conditions for Joint No. 3 are more severe than the flood conditions, it will be applied in connection with the former. The formula developed in the theory to be used for this purpose is, since

$$u = \frac{l}{3}$$

$$\text{Max } p = \frac{2W}{l}\left(2 - \frac{3u}{l}\right) = \frac{2W}{l} = \frac{1676 \times 7}{3 \times 16 \times 40} = 6.1 \text{ tons per sq.ft.},$$

which shows that at the toe of the dam, 50 feet below the top, the presence of ice causes a pressure well within the prescribed limit of 20 tons per square foot.

Joint No. 4, Flood Conditions

Using the same equations (for l, Series C, Stage II) to determine l, A, and y, but with the following values for the variable factors, $A_0 = 1426$ square feet, $H = 60$ feet, $h = 10$ feet, $l_0 = 29.5$ feet, and $y_0 = 12$ feet, we find that

$l = 35$ feet, $A = 1749$ square feet, and $y = 12.7$ feet.

Joint No. 4, Ice Conditions

The same equations (Series D for l) will be employed here as in determining the values at Joint No. 3, with the following values: $A_0 = 1676$ square feet, $H_1 = 50$ feet, $h = 10$ feet, $l_0 = 40$ feet, and $y_0 = 14.5$ feet.

These give $l = 45$ feet, $A = 2101$ square feet, $y = 15.9$ feet, and $p = 6.8$ tons per square foot.

Joint No. 5, Flood Conditions

With the same equations for l, (Series C) A, y, and p, the values of the variables in which have become; $A_0 = 1749$ square feet, $H = 70$ feet, $h = 10$ feet, $l_0 = 35$ feet, and $y_0 = 12.7$ feet, we obtain $l = 42.2$ feet, $A = 2135$ square feet, $y = 13.9$ feet, and $p = 7.4$ tons per square foot.

The value of $y = 13.9$ feet, and of $l = 42.2$ feet, establishes the fact that at this point, since $\frac{l}{3} = 14.1$ feet, the resultant pressure for reservoir empty has passed outside the middle third by 0.2 foot. It would seem proper, therefore, to determine the value of q, the maximum pressure at the heel, to see if the limit of pressure has been exceeded, due to this excursion of the resultant beyond the middle third, and the formula to be employed would be

$$q = \frac{2W}{3y},$$

which is a modification of

$$p = \frac{2W}{3u}.$$

Using the former equation and substituting the appropriate values, we find

$$q = 7.46 \text{ tons per square foot,}$$

which is well within the limits prescribed.

Joint No. 5, Ice Conditions

Repeating the use of the equations (Stage II Series D, for l) applied in solving for Joint No. 4, and using the values; $A_0 = 2101$ square feet, $H_1 = 60$ feet, $h = 10$ feet, $l_0 = 45$ feet, and $y_0 = 15.9$ feet, we obtain $l = 50.6$ feet, $A = 2579$ square feet, $y = 17.5$ feet, and $p = 7.4$ tons per square foot.

It should be noted here that while the flood conditions gave a value to y for Joint No. 5, which showed that the resultant, reservoir empty, fell outside the middle third, the ice conditions indicate that the resultant lies within the middle third 0.7 feet, since $\frac{l}{3} = 16.8$ feet, and $y = 17.5$ feet. Under these circumstances then, the flood conditions control, and make it necessary to batter the back, while the ice conditions do not. This will make it necessary to use the equations coming under Stage III for the flood conditions, where the value of $y = \frac{l}{3}$ is assigned, while equations under Stage II will be used for ice conditions.

Joint No. 6, Flood Conditions

The following equation under Stage III, Series C, must be employed:

$$\left(1 - \frac{cH}{\Delta h}\right) l^2 + \left(\frac{2A_0}{h} + l_0\right) l = \frac{H^3}{\Delta h}.$$

In the above expression the factors take the following values. Note that h is here made 20 feet. $A_0 = 2135$ square feet, $H = 90$ feet, $h = 20$ feet, $l_0 = 42.2$ feet.

Substituting these values in the above, completing the square, and solving for l, we obtain

$$l = 66 \text{ feet.}$$

while A is found equal to 3217 square feet, and

$$\text{Max. } p = 7.1 \text{ tons per square foot.}$$

It is, of course, unnecessary to determine y here, since its value has been established as $\frac{l}{3}$.

It is necessary, however, to determine the value of t, which represents the amount the back face has to be battered in going from Joint No. 5 to Joint No. 6. Using the following equation, in which all the quantities are known, we have

$$t = \frac{2A_0(l - 3y_0) - hl_0^2}{6A_0 + h(2l_0 + l)} = 4.3 \text{ feet.}$$

Joint No. 6, Ice Conditions

We will again use the equations employed in solving for the quantities under Joint No. 5, using the values $A_0 = 2579$ square feet, $H_1 = 80$ feet, $h = 20$ feet, $l_0 = 50.6$ feet, and $y_0 = 17.5$ feet.

There follows from these values $l = 62.4$ feet, $A = 3709$ square feet, $y = 20.8$ feet and $p = 8.7$ tons per square foot.

Joint No. 7, Flood Conditions

Using the same equations for Joint No. 7 that were employed for Joint No. 6, but increasing the value of h to 30 feet, we obtain from the following values of the factors, $A_0 = 3217$ square feet, $H = 120$ feet, $h = 30$ feet, $l_0 = 66$ feet, the quantities, $l = 92.5$ feet, $A = 5595$ square feet, $p = 8.8$ tons per square foot and $t = 1.5$ feet.

Joint No. 7, Ice Conditions

It will be noted that at Joint No. 6, the value of $y = 20.8$ feet is just equal to $\frac{l}{3} = \frac{62.4}{3} = 20.8$ feet.

In consequence of this, it will be necessary to take a value of y for all successive joints equal to $\frac{l}{3}$, to prevent the resultant, reservoir empty, from passing beyond the middle third. This compels the use of Stage III, under Series D, which will produce a batter in the back face, and the following equation will therefore be employed:

$$\left(1 - \frac{cH_1}{\Delta h}\right) l^2 + \left(\frac{2A_0}{h} + l_0\right) l = \frac{H_1^3 + 6TH_1}{h\Delta}.$$

Here $A_0 = 3709$ square feet, $H_1 = 110$ feet, $h = 30$ feet, $l_0 = 62.4$ feet, which, if substituted in the above, give

$$l = 85.6 \text{ feet},$$

while $A = 5929$ square feet, $p = 10.1$ tons per square foot, and $t = 2.0$ feet.

Down to Joint No. 7, two cross-sections have now been developed side by side, one for Flood Conditions and one

for Ice Pressure, but it will be noted by reference to the table of results, and to the profile, that to and including Joint No. 5, the length of each joint under Ice Pressure has exceeded the length under Flood Conditions, and that at Joint No. 6, the difference is only slightly in favor of the Flood Condition profile. As a consequence, down to Joint No. 6 the section that must be used is that established by Ice Pressure formulæ. We must, therefore, in examining Joint No. 7, for Flood Conditions, use values for area, weight, etc., which represent the values of the Ice Pressure profile, and not values from the Flood Level profile. This in effect amounts to investigating Joint No. 7 and its successors, whose length has been determined by Ice Pressure formulæ, by the equations applying under Flood Level Conditions, and where the latter develops a length of joint in excess of that determined by Ice Pressure the greater length will be used.

We will first complete the cross-section under Ice Pressure by examining Joints No. 8, 9, and 10, each of which is 30 feet below the next above, and in each of which the same formulæ apply as in Joint No. 7. The results alone are given.

At joint No. 8, Ice Conditions	$l = 108.6$ ft.,	$A = 8{,}842$ sq. ft.,	$p = 11.9$ tons,	$t = 1.2$ ft.
" " " 9, " "	$l = 131.6$ "	$A = 12{,}445$ "	" $p = 13.8$ "	$t = 0.8$ "
" " " 10, " "	$l = 155.1$ "	$A = 16{,}745$ "	" $p = 15.7$ "	$t = 0.8$ "

Joint No. 7, Flood Conditions Combined with Ice Pressure Profile above Joint No. 6

Proceeding now to investigate Joint No. 7, under Flood Conditions, with the Ice Pressure profile above Joint No. 6 providing the values of the factors to be used, and

employing the formulæ under Stage III, Series C, we find, using $A_0 = 3709$ square feet, $H = 120$ feet, $h = 30$ feet, and $l_0 = 62.4$ feet,

$$l = 82.9 \text{ feet}, \qquad A = 5889 \text{ square feet},$$

$$p = 10.3 \text{ tons per square foot} \qquad \text{and} \qquad t = 1.2 \text{ ft.}$$

But a comparison of these results with those obtained for the same joint under Ice Pressure shows that the latter, being larger, controls the length of joint.

Joint No. 8, Flood Conditions Combined with Ice Pressure Profile above Joint No. 7

Similarly, for Joint No. 8, we must employ the factors resulting from the Ice Pressure profile in the solution of the quantities in the Flood Conditions. These factors become $A_0 = 5929$ square feet, $H = 150$ feet, $h = 30$ feet, and $l_0 = 85.6$ feet.

From which we derive, by the use of the same formulæ employed under Joint No. 7, the following: $l = 111.2$ feet, $A = 8881$ square feet, $p = 11.6$ tons per square foot, and $t = 1.9$ feet.

Here we find the above quantities exceeding in value those determined by the Ice Pressure formulæ, so that the former must be employed. In other words, $l = 111.2$ feet is used in the profile instead of $l = 108.6$.

Joint No. 9, Flood Conditions with Ice Pressure Profile Combined

The factors to be used are those just derived, $A_0 = 8881$ square feet, $H = 180$ feet, $h = 30$ feet, and $l_0 = 111.2$ feet.

From which we obtain with the same formulæ, $l = 138$ feet, $A = 12{,}619$ square feet, $p = 13.3$ tons per square foot, and $t = 1.6$.

Here again the values of the above quantities are greater than those determined by Ice Pressure formulæ, so they must be used.

Joint No. 10, Flood Conditions with Ice Pressure Profile Combined

The factors are $A_0 = 12{,}619$ square feet, $H = 220$ feet, $h = 30$ feet, and $l_0 = 138$ feet; whence, $l = 161.9$ feet, $A = 17{,}119$ square feet, $p = 15.4$ tons per square foot, and $t = 0.3$ foot, which values are again in excess of those obtained from the formulæ for Ice Pressure.

In order to check the above calculations by the graphical method, there is submitted in tabular form

Quantites for Graphic Solution

		Water Units.	Water Units in Tons.*
W ..... =	$17{,}119 \times \frac{7}{3}$ =	40,000	1250
Water Pressure ..... =	$\frac{210^2}{2}$ =	22,050	705
Upward Water Pressure ..... =	$\frac{162 \times 210}{2} \times \frac{2}{3}$ =	11,340	354
Ice Pressure ..... =	$\frac{47{,}000}{62.5}$ =	752	23.5
Resultant			1130

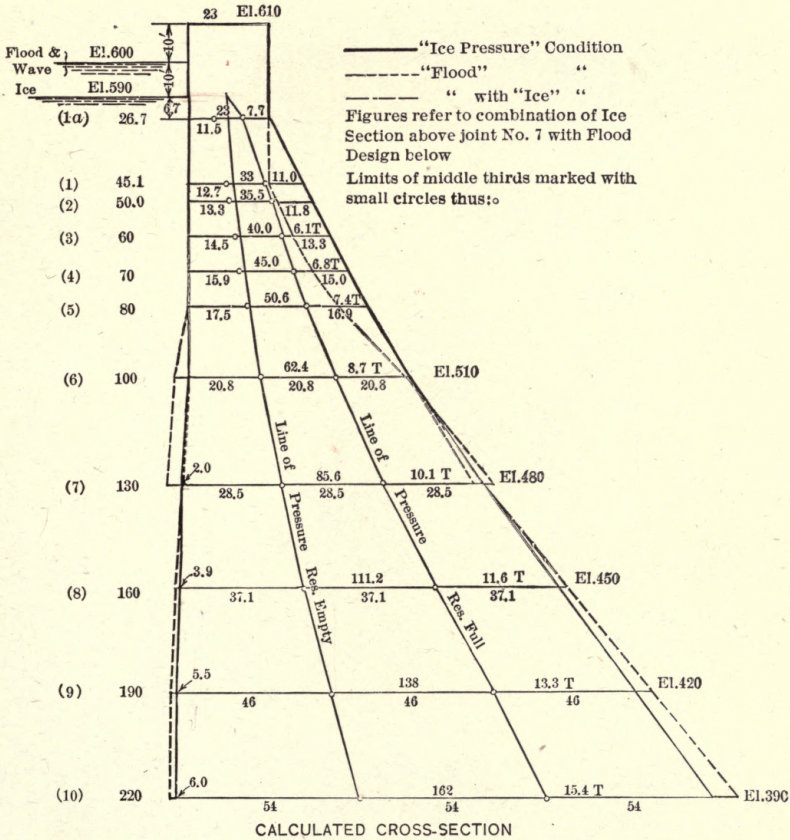
* Multiply 62.5 into Water Units to get pounds, or by .03125 for tons.

Table of Results

(Maximum Flood Level at Elev. 600). $a = 10$ ft. for Flood; $a = 20$ ft. for Ice.

Joint. Ice.	Joint. Flood.	H	h	$H+a$	A_0	A	l	l_0	y	y_0	u	t	Tons per Sq.ft. p	Tons per Sq.ft. q	l_0^2	$A_0 y_0$	H^3
		Ft.	Ft.	Ft.	Sq.ft.	Sq.ft.	Ft.	Ft.	Ft.	Ft.	Ft.	Ft.					
1a	..	6.7	26.7	26.7	.....	615	23.0	23.0	11.5								
	1	35.1	45.1	45.1	.....	1,035	23.0	23.0	11.5	....	$l/3$	0	....	....	529	.....	43,243
1	..	25.1	18.4	45.1	615	1,130	33.0	23.0	12.7	11.5	$l/3$	0	....	....	529	7,075	15,813
	2	40	4.9	50	1,035	1,153	25.0	23.0	11.6	11.5	$l/3$	0	....	....	529	11,900	64,000
2	..	30	4.9	50	1,130	1,298	35.5	33.0	13.3	12.7	$l/3$	0	....	....	1,090	14,375	27,000
	3	50	10	60	1,153	1,426	29.5	25.0	12.0	11.6	$l/3$	0	....	....	625	13,317	125,000
3	..	40	10	60	1,298	1,676	40.0	35.5	14.5	13.3	$l/3$	0	6.1	....	1,260	17,250	64,000
	4	60	10	70	1,426	1,749	35.0	29.5	12.7	12.0	$l/3$	0	....	....	870	17,033	216,000
4	..	50	10	70	1,676	2,101	45.0	40.0	15.9	14.5	$l/3$	0	6.8	....	1,600	24,383	125,000
	5	70	10	80	1,749	2,135	42.2	35.0	13.9	12.7	$l/3$	0	7.4	7.46	1,225	22,245	343,000
5	..	60	10	80	2,101	2,579	50.6	45.0	17.5	15.9	$l/3$	0	7.4	....	2,025	33,425	216,000
	6	90	20	100	2,135	3,217	66.0	42.2	22.0	13.9	22.0	4.3	7.1	7.1	1,781	.....	729,000
6	..	80	20	100	2,579	3,709	62.4	50.6	20.8	17.5	20.8	0	8.7	....	2,560	45,025	512,000
	7	120	30	130	3,217	5,595	92.5	66.0	30.8	22.0	30.8	1.5	8.8	8.8	4,356	.....	1,728,000
* 7	..	110	30	130	3,709	5,929	85.6	62.4	28.5	20.8	28.5	2.0	10.1	10.1	3,894	77,072	1,331,000
	8	150	30	160	5,595	.....	.....	92.5	....	....	....	...	....	....	8,556	.....	3,375,000
8	..	140	30	160	5,929	8,842	108.6	85.6	36.2	28.5	36.2	1.2	11.9	11.9	7,327	.....	2,744,000
	9	180	30	190	.....	.....	.....	.....	....	....	....	...	....	....	.....	.....	5,832,000
9	..	170	30	190	8,842	12,445	131.6	108.6	43.9	36.2	43.9	0.8	13.8	13.8	11,794	.....	4,913,000
	10	210	30	220	.....	.....	.....	.....	....	....	....	...	....	....	.....	.....	9,261,000
10	..	200	30	220	12,445	16,745	155.0	131.6	51.6	43.9	51.6	0.8	15.7	15.7	17,319	.....	8,000,000
Following Results are for Combination of "Ice" Section above Joint 7 with Flood Level Design below Joint 7.																	
* 7	..	110	30	130	.....	5,929	85.6	.....	28.5	....	28.5	2.0	10.1	10.1			
	8	150	30	160	5,929	8,881	111.2	85.6	37.1	28.5	37.1	1.9	11.6	11.6	7,327	.....	3,375,000
	9	180	30	190	8,881	12,619	138.0	111.2	46.0	37.1	46.0	1.6	13.3	13.3	12,365	.....	5,832,000
	10	210	30	220	12,619	17,119	162.0	138.0	54.0	46.0	54.0	0.3	15.4	15.4	19,044	.....	9,261,000

The finally adopted cross-sections of the Olive Bridge Dam and the Kensico Dam, in addition to being investigated by means of the formulæ of Chapter IV, and others



NOTE.—Earth on down-stream toe would not increase compressive stress beyond 20 T. per square foot, if taken at 100 pounds per cubic foot.

FIG. 16.

like them, taking into account variation in distribution of uplift, were subjected to studies based on the method of Ottley and Brightmore, given in Chapter VIII, for determining the shearing stresses to be expected on *vertical*

planes. This maximum shear intensity was found to occur within 8 or 9 feet of the down-stream edge of the base, in each case and did not exceed 92 pounds per square

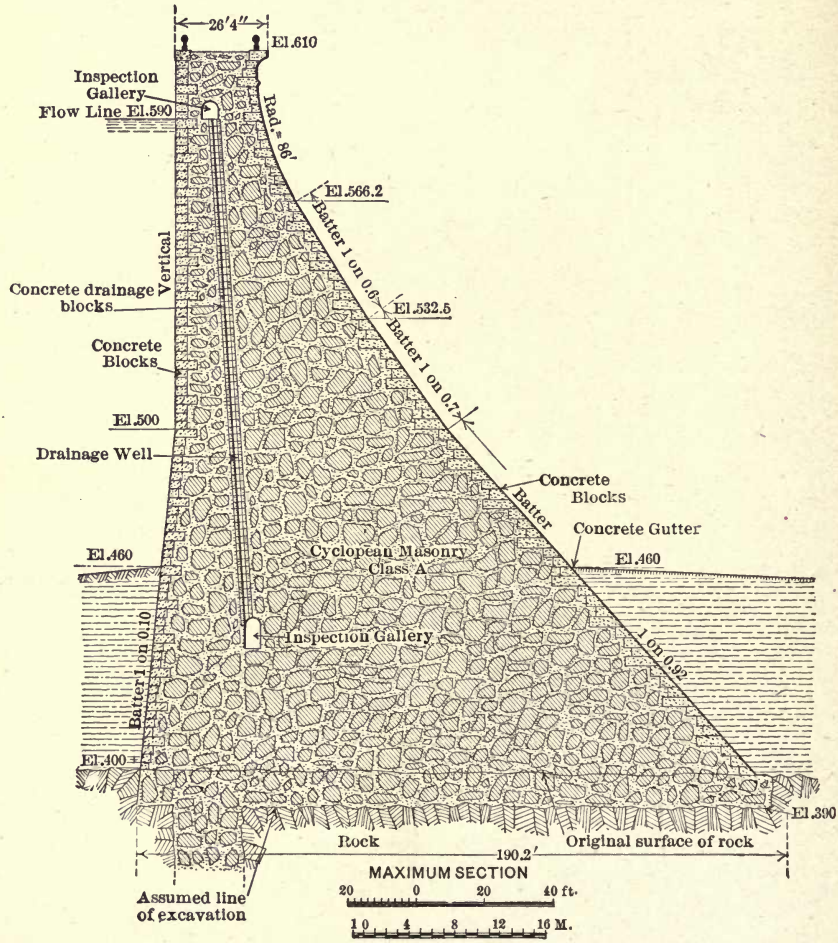


FIG. 17.—Olive Bridge Dam.

inch. This was with the assumption that the base (at the foot of foundation excavation) made an acute angle with the down-stream face. The shear, of course, was

zero at the up-stream and down-stream edges of the base. Incidentally, the assumed masonry density was checked with the final design, allowing for all openings within the structure.

Besides these investigations special studies were under-

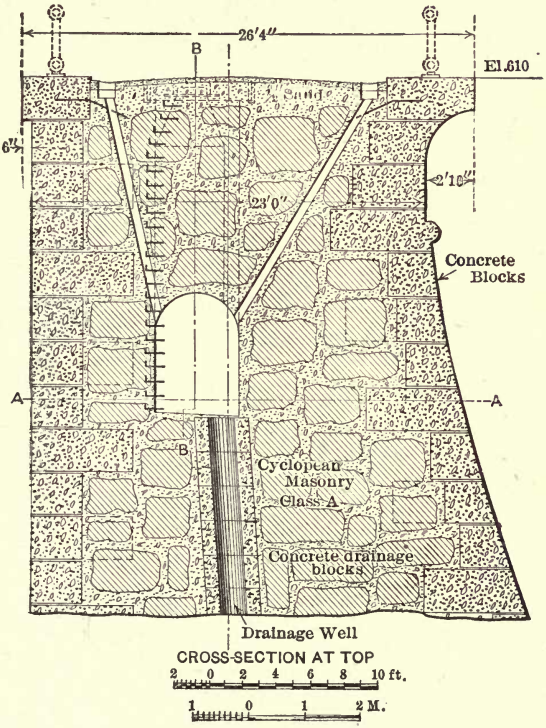


FIG. 18.

taken to ascertain the probable effect on the stresses, of the large, stream flow passageway in the lower part of the Olive Bridge Dam left open during construction and subsequently closed with concrete; also the effect, in the case of the Kensico Dam, of the lower portion's cracking longitudinally, due to temperature changes during construc-

tion. This cracking was assumed to take place so as to divide the lower portion into three parts that would serve as three lines of huge, supporting blocks for the portion above. The stresses that would probably result in each case were found to give no cause for concern.

CHAPTER VI

WEIR OR OVERFALL TYPE OF DAM

In the following pages the development of the cross-section for the overfall, spillway, or weir type of dam will be considered, the function of which type is to permit the flow of water over the top, or "crest." This class of structure may serve primarily either of two purposes:

(1) To make accurate measurements of the discharge over the crest, the weir being, in that case, called a "measuring weir."

(2) To allow surplus water to escape from the side of a canal or reservoir, the weir being then styled a "waste weir" or "spillway."

The same structure may, however, serve the double purpose at one and the same time, as, for example, where the flow over a spillway dam is gaged.

In the first case, the discharge capacity per unit length of weir crest is known in terms of the depth or head of water on the crest, while in the second case, the discharge, usually a maximum, is either known or assumed, and the length of weir necessary for a given head or allowable range in head is thereby determined.

Among the many proposed, the simplest form of expression for weir discharge is the Francis formula,

which shows the relation among those factors entering into the evaluation of the discharge, as follows:

$$Q = CLH^{3/2},$$

in which

Q = volume of discharge per unit of time;
C = an empirical coefficient;
L = length of weir (corrected for end contractions, if any, of the issuing sheet);
H = head on the crest corrected for the effect of velocity of approach.

It is unnecessary to give here more than passing consideration to the subject of the hydraulics of weirs in its various phases; the reader is referred for fuller discussion to almost any work on hydraulics, but especially to weir experimentation.*

As it is impossible to predetermine exactly the discharge that is to pass over a waste weir and as allowance must always be made for unusual storms or floods in providing a length of crest in any given case, a knowledge of the precise discharge capacity of the waste weir will be of less importance than in the case of a measuring weir.

Nevertheless, in addition to the question of spillway length to be provided, there remains the problem of arriving at a proper form of cross-section for the structure. This latter may be determined from a knowledge of the discharge, even though it be inexact, since it leads to approx-

* Merriman's "Treatise on Hydraulics;" Trautwine's "Engineers' Pocketbook;" "Water Supply and Irrigation, Paper 200" on "Weir Experiments, Coefficients and Formulas," Dept. of Interior, U. S. Geol. Survey; "Hydraulics of Rivers, Weirs and Sluices," by David A. Molitor.

imate probable velocities attained by the sheet of water in its fall over the crest together with the shape of the sheet.

One condition to be fulfilled by a spillway dam is that it shall discharge a maximum quantity of water per unit of time for a given length and head on crest, and without endangering the structure in any way. It is evident that for high heads and consequent heavy sheets of water, the best results will obtain with regard to the structure's safety, if the overfall takes place without shock at any point; that is, if the structure is fitted smoothly to the shape that the sheet would naturally assume in flowing over the crest, and if the cross-section at the bottom, downstream, is of such form and material as to lead the water away from the structure, without impact or erosion at the toe.

It has been observed that a discharge over a spillway dam has produced vibrations that may affect the stability of the structure. These may be accounted for as follows:

If the sheet of falling water leaves the dam's face and then impinges upon it lower down, air will be entrained in the intervening space that will be gradually exhausted by the rapidly moving filaments of the adjacent sheet. As the condition of a vacuum is approached the superior atmospheric pressure deflects the entire sheet of water violently against the face of the dam, causing a shock, provided the mass of falling water is not too great to resist such deflection. The atmospheric pressure, acting also on the masonry of the dam, will tend to force it downstream, if not of sufficient mass, during maintenance of a vacuum under the sheet. This is the so-called " suction "

down-stream exerted on the masonry. Repeating cycles of first entraining air, then exhaustion, and then readjustment, coupled with the consequent changes in the impinging sheet lower down the face, produce the vibrations above referred to.

The danger from these lies in the possibility of action in an up and down-stream direction and of acting sympathetically with the structure's rate of vibration, in which case the cumulative effect might lead to the ultimate destruction of the dam.

The smooth face fitted to the curve of fall would prevent the water suddenly leaving it at any point. Furthermore, the insurance against the formation of a vacuum between the face of the dam and the water sheet may be accomplished by so proportioning the face of the dam that the cross-section would extend well into the water sheet throughout its entire extent.

Fanning * recommended a down-stream face of the weir, "slightly more full than the parabolic curve which the film of water at two-thirds depth on the crest tends to take," in order that the overflowing water might not lose contact with the masonry at any point. The foot of this curve should be joined with the river bed by a vertical curve of approximately 100 feet radius,† (Fanning) tangent to the face curve and to the river bed. He further indicated how the above face curve, thus forming what is known as the *ogee* cross-section for high weirs, may be

* "A Treatise on Hydraulic and Water Supply Engineering," Ed. 1899, by J. T. Fanning.

† For high dams. This radius would be far too high a value for moderately high dams. The slope of the face and valley downstream would regulate the radius at toe to a great extent.

resolved into steps, for the purpose of breaking the fall of water into a number of smaller falls. The steps "kill" acceleration of the falling sheet. It is obvious that the force of the falling water in such modified section is constrained to act in a *vertical* direction. The "steps"

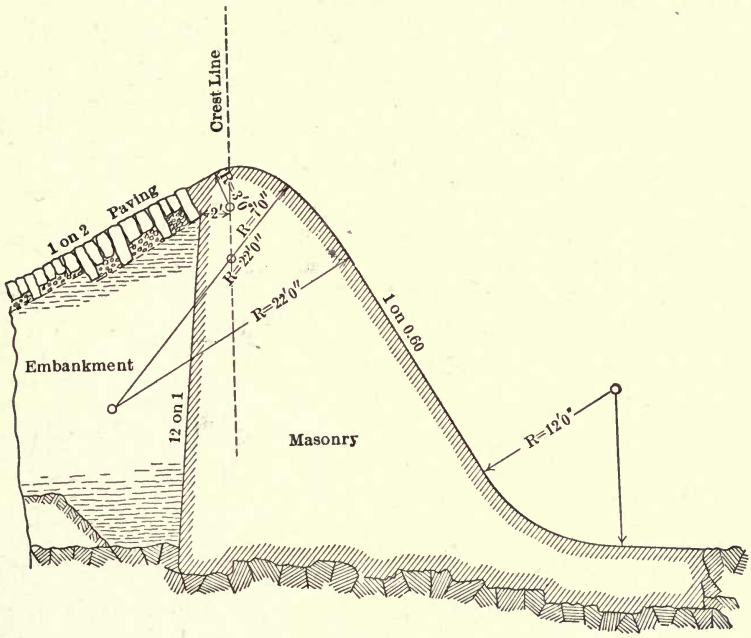


FIG. 19.

should be so proportioned that they will project well into the under surface of the sheet. Figures 19 and 20 illustrate, respectively, these ogee and stepped types and are cross-sections of recent structures built for heads not much over 5 feet. It may be noted that the tops are smooth curves in both types of section, that is, stepping should not occur until the sheet is well over the crest. The stepped type serves where high velocities at the down-

stream toe are objectionable. In the matter of construction, face stones should be set on edge radially with respect to the curves of the face, especially at the foot. In a concrete structure, hard stone is often thus embedded at this lower portion to resist erosion.

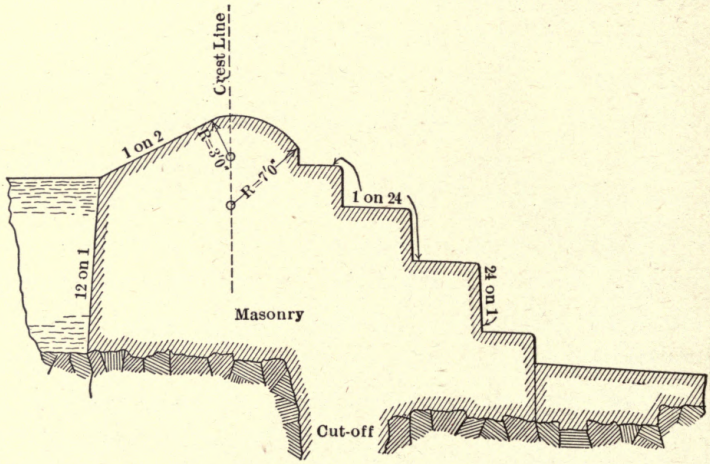


FIG. 20.

A parabola can be determined to fit the shape of the falling sheet and extend just inside its lower surface, or *nappe*, as the French term it.

First. General formulæ for design Eqs. (2) and (3) and Eqs. (9) and (10), will be derived, as has been done for the gravity dam, in Chapter III, and

Second. Consideration will be given to the shape of the nappes and the determination of a value for the parameter of the parabola, appearing in the general formulæ as a factor.

Formulæ for Design.

Spillway Dam.

Derivation.—In Series F, Formulæ for Design, Stage II, page 67, there appears an expression by which a trapezoidal cross-section for a spillway dam may be calculated. From the preceding discussion it is evident that the manner of determining the shape at or near the top requires a more detailed treatment; but the formulæ of Series F are sufficient for fixing the cross-section lower down.

One method would be to round the top of the trapezoidal section in a practical way to suit the case in hand after comparison with crests of approved existing structures, knowing their discharge capabilities. The structures of Figs. 19 and 20 were designed according to this method.

In the following, however, a general parabolic section is derived, corresponding to Stage I, or the rectangular section, of the series of formulæ for design heretofore given and following the identical principles.

As the shape of the parabolic section is fixed by the falling sheet of water, there being no possibility of ice thrust during overfall, and as ice thrust near the top, with no overfall, would have to be resisted by proper reinforcement, vertically, near the up-stream face, it would seem reasonable to ignore that feature in the formulæ for design. But, as the extent downwards from the crest of the parabolic section would be affected by such thrust near the crest, it is thought desirable to include this factor, for the purpose of investigating its effect, if for no other reason. Formulæ, Eqs. (9) and (10), containing the factor T, with water surface at or below the crest, will

be given, therefore, immediately after those first derived for overfall conditions.

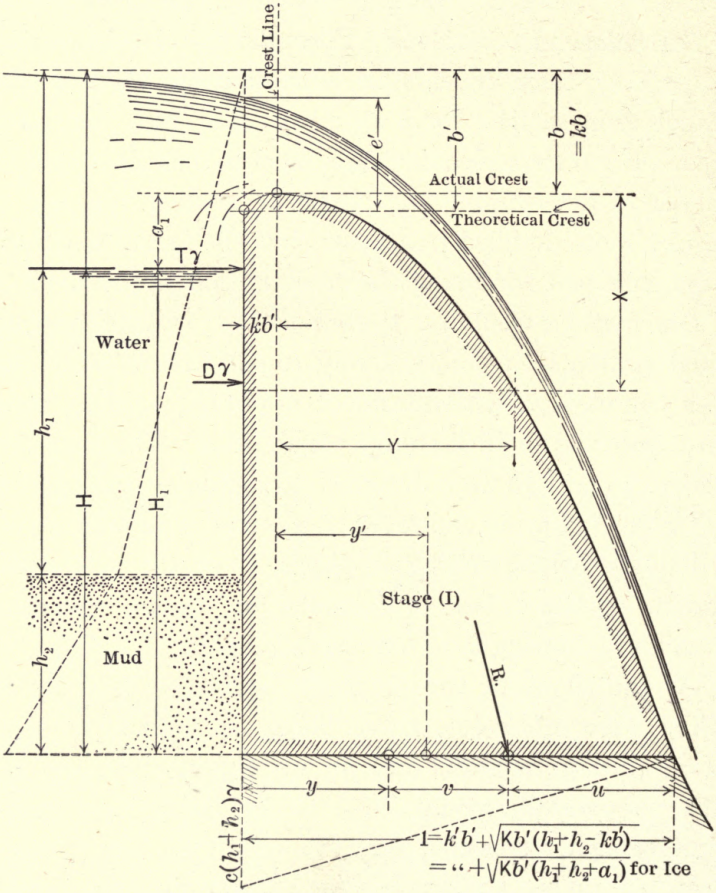


FIG. 21.

As before, the middle third limit, for center of resultant pressure on any horizontal joint, will be retained.

Likewise, for any given case, the terms or factors expressing other than the imposed conditions of loading must be equated to zero.

Cases may arise where the highest head of water overtopping the dam does not provide the critical stage for stability, due to a rapidly rising backwater with the increased discharge. Such must be especially investigated for determination of the critical head.

In addition to the nomenclature given in Chapter III, p. 31, et. seq., the following designations, in connection with Fig. 21, will be employed:

b' = vertical distance from the water surface in the reservoir down to the *theoretical* crest of spillway when the crest is overtopped.

kb' = vertical distance from the water surface down to the *actual* crest of spillway when overtopped. ($kb' = b$, of Chap. III.)

$$k = 0.888 \text{ (Bazin).}$$

$k'b'$ = horizontal distance between the vertical line through the crest (crest line) and the up-stream vertical face.

$$k' = 0.25 \text{ (Bazin).}$$

X = any abscissa (vertical) of the down-stream face of spillway.

Y = ordinate (horizontal), corresponding to X, origin of rectangular co-ordinates being at the actual crest.

$Y^2 = Kb'X$, equation of the parabolic, down-stream face of the spillway cross-section, with respect to the vertical, X axis and the horizontal, Y axis through the actual crest. (Kb' is the parameter of the parabola.)

K = the constant (considered later in this chapter) determining the parabolic face, above, so that the

curve shall extend within the falling sheet of water.

$K = 2.25$ has been used.*

y' = the ordinate to the centroid of the parabolic segment of the cross-section above any joint at distance X below the origin.

$$y' = \tfrac{3}{8}Y.$$

Area of segment $= \frac{2}{3}XY$, down to any level X below the origin.

Overfall, or Flood Level Design (No Ice).

In connection with the following, see Fig. 21.

Overturning moment due to:

(*a*) *Horizontal static water pressure on back* (head $= h_1$).

(*b*) *Upward water pressure on base;* pressure intensity decreasing uniformly from $cH\gamma$ or $c(h_1+h_2)\gamma$, at heel to zero intensity at toe.

(*c*) *Mud (liquid) pressure on back* (head h_2) as before.

(*d*) *Dynamic pressure of water,* $D\gamma$.

(*e*) *Water flowing over top of dam,* weight of water of depth b, on top of dam being neglected.

For condition of no dynamic pressure, $D = 0$.

For condition of no upward water pressure, $c = 0$.

For condition of no mud (i.e., mud being replaced by water) make $h_2 = 0$, $h_1 = H$.

As in the previous deductions, the fundamental equation for length of joint, l, is

$$l = u + v + y,$$

or

$$l = u + \frac{\frac{M}{\Delta\gamma}}{A} + y,$$

* Mr. Richard Muller in "Engineering Record" of Oct. 24, 1908.

in which

$$\frac{M}{\Delta\gamma}=(h_1+h_2)[h_1^2-(kb')^2]\frac{1}{2\Delta}-\frac{h_1^3-(kb')^3}{3\Delta}+\frac{h_2^2}{2\Delta}\left(h_1+h_2\frac{\gamma'}{3\gamma}\right)$$
$$+\frac{D}{2\Delta}(2h_2+h_1-kb')$$
$$+\left[\frac{c(h_1+h_2)l^2}{6\Delta}=\frac{c(h_1+h_2)}{6\Delta}\left(k'b'+\sqrt{Kb'(h_1+h_2-kb')}\right)^2\right].$$

In deducing the expression for A, or the total area of the cross-section above the joint l, the top will be considered horizontal up-stream from the crest line.

$$A=k'b'(h_1+h_2-kb')+\tfrac{2}{3}(h_1+h_2-kb')\sqrt{Kb'(h_1+h_2-kb')}$$

or

$$A=\tfrac{1}{3}(h_1+h_2-kb')(k'b'+2l),$$

$$u=\frac{l}{3}\quad\text{and}\quad l=k'b'+\sqrt{Kb'(h_1+h_2-kb')}.$$

(This expression for l may be more conveniently held for the last substitution in derivation.)

y may be derived by taking static moments about the up-stream edge of the joint l, as follows:

$$y=\frac{\frac{(k'b')^2}{2}(h_1+h_2-kb')+\frac{2}{3}(h_1+h_2-kb')(l-k'b')[\frac{3}{8}(l-k'b')+k'b']}{k'b'(h_1+h_2-kb')+\frac{2}{3}(h_1+h_2-kb')(l-k'b')}$$

whence

$$y=\frac{k'b'}{4}+\frac{3}{4}\frac{l^2}{(k'b'+2l)}=\frac{k'b'(k'b'+2l)+3l^2}{4(k'b'+2l)}.$$

Substituting the above values for u, y, $\frac{M}{\Delta\gamma}$ and A in the fundamental equation for l, and reducing, gives

$$-6h_1^3+(h_1+h_2)[2k'b'l\Delta+(7\Delta-6c)l^2+18(kb')^2-3(k'b')^2\Delta$$

$$-18(D+h_1h_2)]=\frac{6h_2^3\gamma'}{\gamma}+18Dh_2+kb'[2k'b'l\Delta+7l^2\Delta$$

$$+12(kb')^2-18D-3(k'b')^2\Delta].$$

General Formula:

By substituting in this last expression values of l and l^2 involving K, as suggested above, and reducing, gives

$$-6h_1^3+(h_1+h_2)^2(7\Delta-6c)Kb'$$

$$+(h_1+h_2)\{2k'b'[(8\Delta-6c)\sqrt{Kb'(h_1+h_2-kb')}$$

$$+3(\Delta-c)(k'b')]-2k(b')^2[(7\Delta-3c)K-9k]$$

$$-18(D+h_1h_2)\}-16kk'(b')^2\Delta\sqrt{Kb'(h_1+h_2-kb')}=\frac{6h_2^3\gamma'}{\gamma}$$

$$+18Dh_2+kb'[(6k'b')^2\Delta-k(b')^2(7K\Delta-12k)-18D] \quad . \quad (1)$$

From this General Formula, Eq. (1), there may be derived directly Eqs. (2) and (3), formulæ for the parabolic cross-section corresponding to those for the rectangular cross-section presented in "Series F, Stage I, Cases (1) and (2)" of Chapter III, p. 66.

Case (1). Condition $h_1=H$; $h_2=0$, or no mud, H, to be determined, *underlined* in Eq. (2).

$$6\underline{H}^3+\underline{H}^2(6c-7\Delta)Kb'$$

$$+\underline{H}\{2k'b'[(6c-8\Delta)\sqrt{Kb'(\underline{H}-kb')}+3(c-\Delta)k'b']$$

$$-2k(b')^2[(3c-7\Delta)K+9k]+18D\}$$

$$+16kk'(b')^2\Delta\sqrt{Kb'(\underline{H}-kb')}=kb'[18D-6(k'b')^2\Delta$$

$$-k(b')^2(12k-7K\Delta)] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Case (2). Condition h_1 of known value, h_2, to be determined, *underlined* in Eq. (3).

$$
\begin{aligned}
&6\underline{h_2}^3\left(\frac{\gamma'}{\gamma}\right)+(\underline{h_2}+h_1)^2(6c-7\Delta)Kb' \\
&\quad+(\underline{h_2}+h_1)\{2k'b'[(6c-8\Delta)\sqrt{Kb'(\underline{h_2}+h_1-kb)} \\
&\quad+3(c-\Delta)k'b']-2k(b')^2[(3c-7\Delta)K+9k) \\
&\quad+18(D+\underline{h_2}h_1)\}+18\underline{h_2}D+16kk'(b')^2\Delta\sqrt{Kb'(\underline{h_2}+h_1-kb')} \\
&\quad=kb'[18D-6(k'b')^2\Delta-k(b')^2(12k-7K\Delta)]-6h_1^3, \quad . \quad . \quad (3)
\end{aligned}
$$

It will be noted that the unknowns, in Eqs. (2) and (3) above, are H and h_2, respectively, and that the known terms or factors are recurrent, and in any design need be substituted but once.

A convenient method of using either of the above Eqs. (2) or (3) is as follows:

(1) Substitute chosen values for the known terms, according to assumptions and conditions, and reduce. The right-hand member of either of the equations reduces to a number, the left-hand member (an expression of the third degree) to terms involving the unknown (H or h_2) and numerical factors.

(2) Substitute at least three successive trial values for the unknown (these values always positive) and compute the corresponding values of the left-hand side of the equation.

(3) Plot a curve with these trial values for the unknown as ordinates and the values of the left-hand member corresponding as abscissæ. The ordinate corresponding to the predetermined value of the *right-hand* member will at once yield the correct value of the unknown.

Illustration of Use of Equations. Condition.—*Hydrostatic water pressure on back, with overtopping.* Eliminating all but hydrostatic and overtopping factors in Eq. (2), above, gives, as a result, Eq. (4), following:

$$\begin{aligned}&6H^3-7Kb'H^2\Delta-H\{2k'b'\Delta[8\sqrt{Kb'(H-kb')}+3k'b']\\&\quad-2k(b')^2(7K\Delta-9k)\}+16kk'(b')^2\Delta\sqrt{Kb'(H-kb')}\\&\quad=k(b')^3[k(7K\Delta-12k)-6(k')^2\Delta]. \qquad (4)\end{aligned}$$

Eq. (4) will serve to illustrate the foregoing remarks, using the numerical coefficients for k, k' and K, respectively, 0.89, say; 0.25; and 2.25. Substituting these values in Eq. (4) and reducing, gives an expression in terms of b', Δ, and H, the last, the unknown to be determined, as follows:

$$\begin{aligned}&H^3-2.625b'H^2\Delta-H\{b'\Delta[\tfrac{2}{3}\sqrt{2.25b'(H-0.89b')}\\&\quad+0.0625b']-(b')^2(4.6725\Delta-2.3763)\}\\&\quad+0.5933(b')^2\Delta\sqrt{2.25b'(H-0.89b')}\\&\quad=(b')^3(2.0236\Delta-1.4099). \qquad (5)\end{aligned}$$

Eq. (5), then, contains Bazin's coefficients * and a constant, fixing the parabolic face that will determine a cross-section presumably acceptable as to flow conditions. The conditions of stability down to the depth of water, H, on the base at that depth, is that of the "middle third limit," for the resultant pressures on the joints at and above that base. The loading conditions, for simplicity, comprise only those of horizontal static water pressure on back and "overtopping."

---

* See Table III.

Assuming a head (b') on the theoretical crest of 20 feet and $\Delta = 2.24$ and inserting these values in Eq. (5) gives

$$H^3 - 117.6H^2 - (29.867\sqrt{45(H-17.8)} - 3139.720)H$$
$$+531.623\sqrt{45(H-17.8)} = 24{,}984. \quad (6)$$

Preliminary to the prosecution of a solution of Eq. (6), it will prove convenient to calculate and plot a curve for the value, $\sqrt{45(H-17.8)}$, for different values of H, ranging from 18 to 100 in this case. The smooth curve resulting can be used in the subsequent assumptions for "trial" H's, to pick off corresponding values of $\sqrt{45(H-17.8)}$ for entry in the left-hand member of Eq. (6). This factor, $\sqrt{45(H-17.8)}$, is useful in indicating at once the lowest positive value that may be assumed for H. In the case here given H cannot have a positive value less than 17.8.

Without reproducing the curves, the results of the trial values of H, assumed, together with the corresponding values of $\sqrt{45(H-17.8)}$ obtained from the preliminary curve are contained in the following table, for the case here under consideration:

Assumed H.	$\sqrt{45(H-17.8)}$.	Left-hand Member of Eq. (6).
95	58.8	−40,325
100	60.8	−11,678
105	62.6	+27,425

From the curve (plotted with the assumed values of H in the first column, and the corresponding numbers of

the third column of the table, as abscissæ and ordinates, respectively) the ordinate, of value $+24{,}984$, of Eq. (6), gave as its abscissa, a value of H equal to 104.7 feet. That is, this dam may be extended as a "parabolic section" down, until the head of water on its base becomes 104.7 feet, with a head of 20 feet on the theoretical crest. The distance below the actual crest would be $H-kb'=104.7-17.8=86.9$ feet.

(It may be said at this point that an investigation for positive roots of Eq. (6), between 18 and 25, for values of H, results, for the left-hand member of that equation, in decreasing values, as H varies from 18 to 25; below 18, the root would have to be negative.

The factor $\sqrt{(H-17.8)45}$ indicates an imaginary quantity at this stage.)

A dam of the above type and subjected to the assumed conditions, would therefore reach the limit of stability contained in the expression "the middle third limit," at nearly 87 feet from its crest.

This result may be checked by use of the formula for investigation, i.e., calculate the position on the base of the center of pressure for 20 feet head on the theoretical crest and base of parabolic section 84.7 feet below that crest. Making the proper eliminations and substitutions to suit the parabolic section in the first formula for investigation of Chapter IV, p. 70, there results:

$$u=l-y-\frac{H^3+(kb')^2(2kb'-3H)}{6A\Delta} \quad . \quad . \quad . \quad (6a)$$

The area A may be expressed in terms of l, as follows:

$$A=\tfrac{1}{3}(H-kb')(k'b'+2l). \quad . \quad . \quad . \quad . \quad (6b)$$

Whence, from (6*a*) and (6*b*), there is obtained:

$$u = l - y - \frac{H^3 + (kb')^2(2kb' - 3H)}{2(H - kb')(kb' + 2l)\Delta}, \quad . \quad . \quad (6c)$$

Substituting in Eq. (6*c*) the values:

$(H - kb') = 86.9$ feet;

$$l = k'b' + \sqrt{Kb'(H - kb')} = 67.5 \text{ feet};$$

$$H = 104.7 \text{ feet};$$

$$\Delta = 2.24;$$

$$y = \frac{k'b'}{4} + \frac{3}{4}\frac{l^2}{(k'b' + 2l)} = 25.66 \text{ feet};$$

$$l - y = 41.84 \text{ feet},$$

there results, after reduction,

$$u = 41.8 - 19.4 = 22.4 \text{ feet}.$$

But $\frac{l}{3} = 22.5$ feet, which is a sufficiently close agreement.

To continue the cross-section below a head of 104.7 feet, or the foot of the parabolic section, recourse must be had to Formulæ of Design, Series F, Stage II, (*b*), or, in case an ice pressure cross-section is being calculated, similar formulæ of Series E, Stage II, et seq., apply.

Ice Pressure Design.

The formulæ now to be derived are similar to those of Series E, Stage I, of Chapter III, with the same conditions governing for that series, as to loading, that is:

(*a*) *Horizontal, static water pressure on back,* (head h_1);

(*b*) *Ice pressure at surface of water,* ($T\gamma$);

(*c*) *Upward water pressure on base*;

(*d*) *Mud* (*liquid*) *pressure on back* (head h_2), commencing at distance h_2 above joint in question.

In the fundamental formula for l, M here differs from the value given in the preceding consideration, in that the overtopping effect and, consequently, dynamic pressure, are here supplanted by the effect of the ice pressure, which will be referred to the top of the dam, in location, by the vertical distance a_1. This reference will in turn change the expressions for A and l, of the parabolic section.

The procedure, otherwise, remains the same as for the overfall condition.

There follow:

$$\frac{M}{\Delta\gamma}=\frac{1}{6\Delta}\left[(h_1+h_2)(3h_1h_2+6\Gamma+cl^2)+h_1^3+\frac{\gamma'}{\gamma}h_2^3\right];$$

$$A=\tfrac{1}{3}(h_1+h_2+a_1)(k'b'+2l);$$

$$y=\frac{k'b'(k'b'+2l)+3l^2}{4(k'b'+2l)};$$

$$u=\frac{l}{3}.$$

Substituting these in $l=u+\dfrac{\frac{M}{\Delta\gamma}}{A}+y$, and reducing, gives:

$$6h_1^3+(h_1+h_2)[18h_1h_2+36\Gamma+3(2c+3\Delta)l^2$$
$$-\Delta(k'b'+2l)(8l-3k'b')]=-\frac{6\gamma'h_2^3}{\gamma}$$
$$-\Delta a_1[9l^2-(k'b'+2l)(8l-3k'b')]. \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

The equation, $l=k'b'+\sqrt{Kb'(h_1+h_2+a_1)}$, applies here. Substituting this expression for l, together with its square in Eq. (7) and reducing, gives Eq. (8), whence two cases,

previously formulated for the overflow condition, follow directly.

Eq. (8) is:

$$6h_1^3+(h_1+h_2)\left\{18h_1h_2\right.$$

$$+36T+6c\left(k'b'+k'b'\sqrt{Kb'(h_1+h_2+a_1)}\right)^2$$

$$-\Delta\left(\frac{a_1}{h_1+h_2}+1\right)[6(k'b')^2+16k'b'\sqrt{Kb'(h_1+h_2+a_1)}$$

$$\left.+7Kb'(h_1+h_2+a_1)]\right\}=-\frac{6\gamma'h_2^3}{\gamma}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

From Eq. (8) may be obtained:

Case (1). Condition, $h_1=H_1$; $h_2=0$, or no mud. H_1 to be determined.

$$6H_1^2+36T+6c\left(k'b'+k'b'\sqrt{Kb'(H_2+a_1)}\right)^2$$

$$-\Delta\left(\frac{a_1}{H_1}+1\right)[6(k'b')^2+16k'b'\sqrt{Kb'(H_1+a_1)}$$

$$+7Kb'(H_1+a_1)]=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Case (2). Condition, h_1 of known value, h_2 to be determined:

$$\frac{6\gamma'h_2^3}{\gamma}+(h_2+h_1)\left\{18h_2h_1+36T\right.$$

$$+6c\left(k'b'+k'b'\sqrt{Kb'(h_2+h_1+a_1)}\right)^2$$

$$-\Delta\left(\frac{a_1}{h_2+h_1}+1\right)[6(k'b')^2+16k'b'\sqrt{Kb'(h_2+h_1+a_1)}$$

$$\left.+7Kb'(h_2+h_1+a_1)]\right\}=-6h_1^3. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

To recapitulate:

Eqs. (2) and (3) of this chapter are the working equations for the overfall design; Eqs. (9) and (10) for the ice-pressure design. To continue design for overflow conditions, use formulæ of Stage II, (*b*), et seq., of Series F, and to continue ice-pressure design, use formulæ of Stage II, et seq., of Series E, of Chapter III.

After a section has been finally determined, it may have fitted to it circular curves of suitable radii, or tangents, to simplify construction.

A radius equal to the maximum head, with center on the crest line, often proves suitable for first trial near the crest, down-stream. Further adjustment of the upper part of the parabolic face curve, just down-stream of the crest, is also often advantageous. Here, a little cutting away of the curve by appropriate curves of longer radii, where the general direction of the lower nappe of the water sheet is nearer the horizontal than lower down, will tend to increase the flow. There is less chance of serious results from slight vacuum formation at this point than further down, and, besides, the tendency to vibration from such a cause would be in a vertical direction, more or less, with no serious consequence to the structure. Care, however, should be taken that the adjusted curves should be smoothly continuous and follow the sweep of the parabolic face, entering the sheet without abrupt change at any point, so that the falling sheet of water would take its further course without violence.

For a low dam, the parabolic curve down the face may be replaced by a tangent to the curve at a point where the water is well on its more vertical course.

The probable path of the sheet of water should always be studied in connection with any design.

If the lower portion of the face is stepped, the study concerns the flow upon and from each step. The parabolic face or ogee dam is of advantage in this respect, as a preliminary study of the stepped cross-section, since it shows the minimum sized shape for stability, upon which the steps may be arranged to suit the flow. In this connection, compare Figs. 19 and 20.

Sufficient masonry should be placed just down-stream of the toe, depending, to great extent, upon local conditions. The least thickness of this masonry, other considerations being equal, may be approximated by ascertaining the probable depth of back-water just below the dam, after the velocity is reduced, beyond the shallower discharging sheet.* The difference in head between the upper surface of the sheet and the level of the water of greater depth further down-stream could cause an uplift if the head becomes active beneath the toe protection. The thickness of the protection masonry should be at least sufficient to balance this head by the weight of this masonry.

The force of the flow from the toe is sometimes broken up by masonry baffles; or the discharge into the still back-water below the dam, the presence of which is provided for the purpose, may be so directed by the curve of the dam face that the same object is attained.

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* In this connection see paper on "The Hydraulic Jump in Open Channel Flow at High Velocity," by Karl R. Kennison, in Proc. Am. Soc. of C. E. for Sept., 1915.

Shape of the Falling Sheet of Water.

Spillway Dam.

A study of the water sheet, besides outlining a procedure, will indicate its possibilities and limitations—limitations to the procedure because the available data are not so satisfactory or complete as might be wished with respect to their application to the purpose in hand.

Most investigators have concentrated almost wholly upon determination of the discharge, the question of the shape of the water-sheet receiving comparatively little attention, except incidentally to the effect of its fluctuations upon the value of the discharge. An attempt at an approximation, only, can be made in this study, which will be based largely upon the results of M. Bazin's researches.*

M. Boussinesq,† in an account of his investigations, in 1887 emphasized the importance of the relation of the shape of the under side of the sheet to the contraction of the sheet at the crest, he making it the basis of a new theory of flow over weirs.

So M. Bazin, with great care, determined the profiles both of the upper surface and the lower surface of the sheet for two sharp-crested weirs, one 3.7 feet high and the other 1.15 feet high for heads varying from 6 to nearly 18 inches.

By reducing the co-ordinates of the curves of the upper

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* Bazin, H. "Expériences nouvelles sur l'écoulement en déversoir, Annales des Ponts et Chaussées, Mémoires et Documents," 1888, 1890. See also translation by Arthur Marichal and J. C. Trautwine, Jr., Proc. Engineers' Club of Philadelphia, Vol. IX, No. 3 and Vol. X, No. 2.

† "Comptes rendus de l'Académie des Sciences," July 4, 1887.

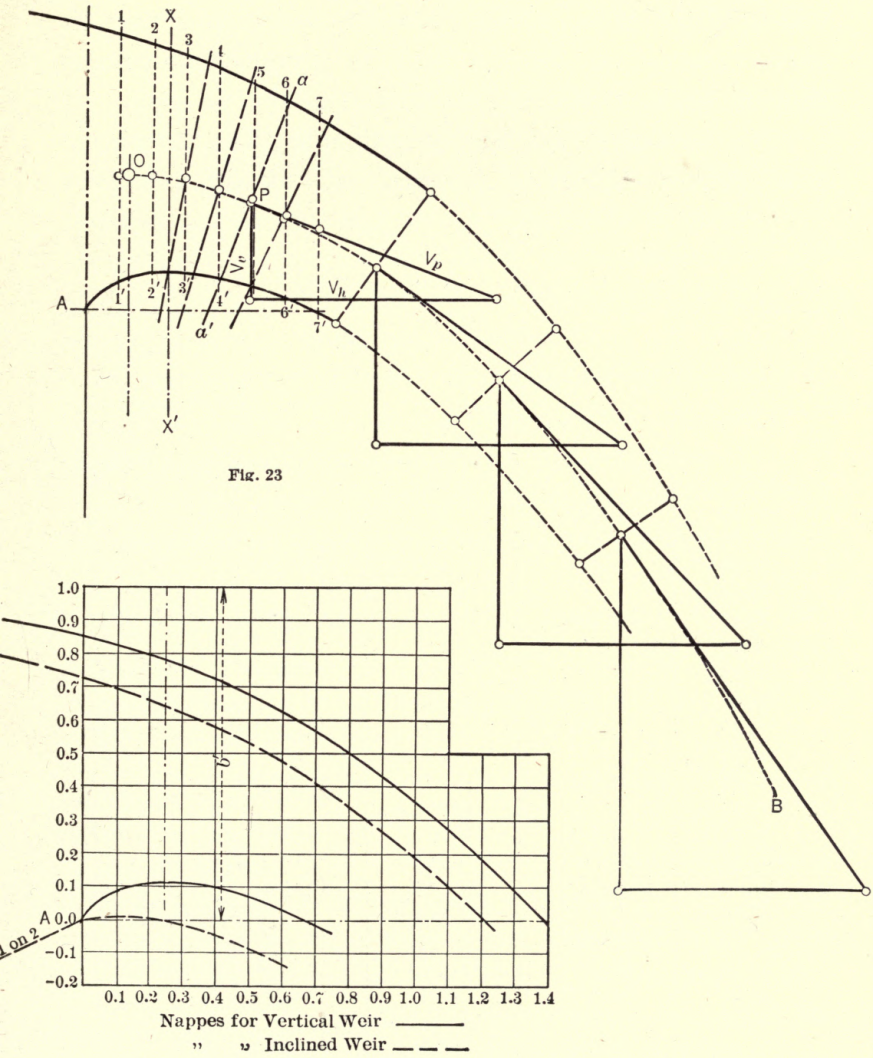

Fig. 23

Fig. 22

FIG. 22 and FIG. 23.

and lower surfaces to a common scale, expressing them as ratios of the head, in each case, he established that for each value of an abscissa there is a corresponding and sensibly constant value of the ordinate.

The axes of rectangular co-ordinates pass through the crest, Figs. 22 and 23, at A, horizontal abscissæ, x, positive

TABLE III

VALUES FOR SHAPES OF NAPPES

Abscissæ $\frac{x}{b'}$	Ordinates $\frac{y}{b'}$			
	Crest Vertical.		Crest Inclined Down-stream.*	
	Upper Nappe.	Lower Nappe.	Upper Nappe.	Lower Nappe.
−3.00	0.997	—		—
−1.00	0.963	—		—
0.00	0.851	0.000	(0.730)	0.000
0.05		0.059		
0.10	0.826	0.085	(0.700)	(0.011)
0.15		0.101		
0.20	0.795	0.109	(0.666)	(0.005)
0.25	(0.778)	0.112		
0.30	0.762	0.111	(0.630)	(−0.014)
0.35		0.106		
0.40	0.724	0.097	(0.585)	(−0.044)
0.45		0.085		
0.50	0.680	0.071	(0.535)	(−0.083)
0.55		0.054		
0.60	0.627	0.035	(0.480)	(−0.130)
0.65		0.013		
0.70	0.569	−0.009	(0.418)	
0.80	(0.507)	(−0.068)	(0.350)	
0.90	(0.437)	(−0.129)	(0.276)	
1.00	(0.360)		(0.196)	
1.10	(0.276)		(0.109)	
1.20	(0.186)		(0.009)	
1.30	(0.085)		(−0.098)	

Numbers in parentheses have been scaled from Bazin's plotted profiles.

* Slope 1 on 2, down-stream. The discharge is increased by nearly 13% over vertical weir's discharge.

to the right, or down-stream, and negative up-stream; the ordinates, y, positive upward from the crest level and negative downward.

These co-ordinates refer only to the curves of the nappe, and not to the face curve of the dam.

Ordinates for the shapes of the nappes as tabulated, likewise are vertical and positive upward; abscissæ, horizontal and positive to the right, both referred to the sharp crest of the weir, or "theoretical crest" of Fig. 21, as origin of rectangular co-ordinates.

The ratios in Table III, if multiplied by the head b', in feet, will give the corresponding locations of points on the respective nappes, in feet, for a given case.

The table gives the values, extended from tabulated values of Bazin from his profiles, and were used in plotting Fig. 22.

The values for the higher weir were found to be very precise, by comparison of 18 different determinations. The final values are here given and are to be multiplied by the head b' for any given case.

The values for a weir whose crest is inclined downstream on a slope of 1 on 2 are also included.

In addition to the shape determinations, the velocity and pressure heads in the sheet, at the contracted section of the sheet, were carefully found experimentally.* The velocities are plotted in Fig. 24 and curves were drawn through them. Their sensible parallelism is significant, indicating simple proportionality for different heads. These

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* M. Bazin, "Annales des Ponts et Chaussées, Mémoires et Documents," 1890.

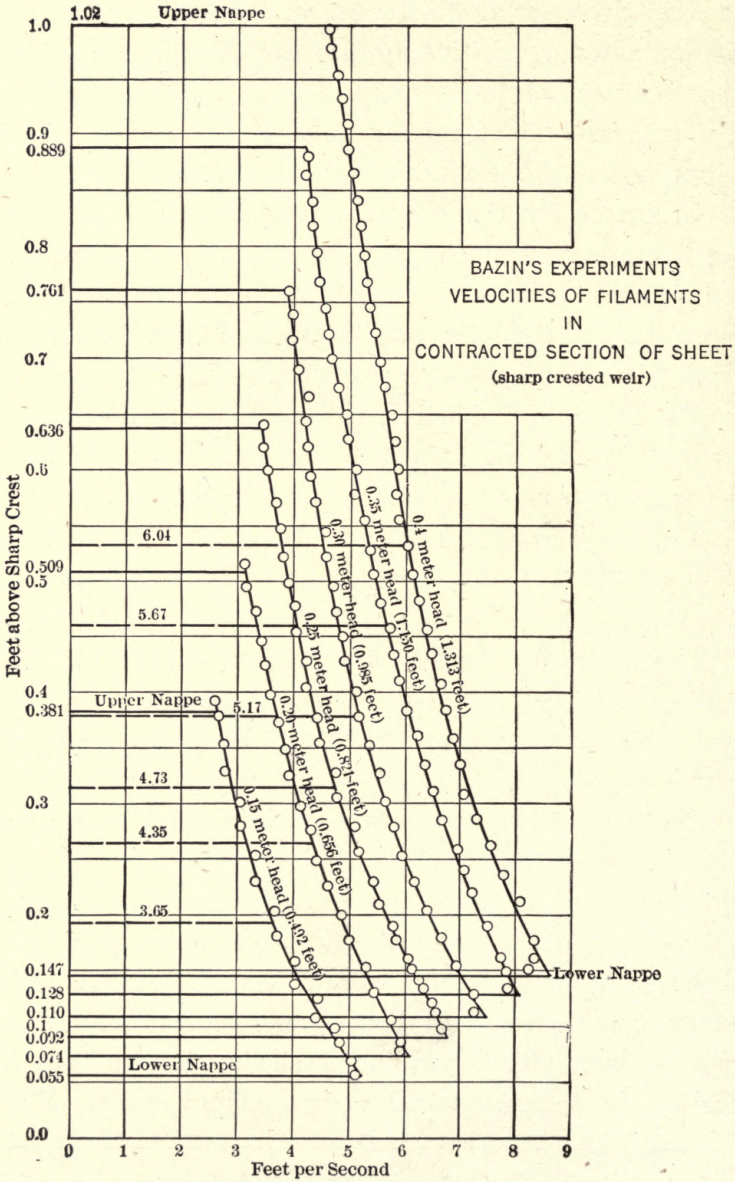

BAZIN'S EXPERIMENTS
VELOCITIES OF FILAMENTS
IN
CONTRACTED SECTION OF SHEET
(sharp crested weir)
Upper Nappe
Lower Nappe
Upper Nappe
Lower Nappe
0.15 meter head (0.492 feet)
0.20 meter head (0.656 feet)
0.25 meter head (0.821 feet)
0.30 meter head (0.985 feet)
0.35 meter head (1.150 feet)
0.4 meter head (1.313 feet)
6.04
5.67
5.17
4.73
4.35
3.65
1.02
0.889
0.761
0.636
0.509
0.381
0.147
0.128
0.110
0.1
0.092
0.074
0.055
Feet above Sharp Crest
Feet per Second

FIG. 24.

velocities were determined for a sharp-crested, vertical weir, 3.7 feet high, with six different heads, ranging from 5.9 to 15.75 inches. As in the preceding experiments, the observed factors were found to vary proportionately with the head.

The purpose of this study is primarily to establish, for the general formulæ of design, preceding, a value for the parameter of the parabolic down-stream face of the weir, encroachment within the water-sheet to be assured. For this purpose the water-sheet's lower surface must be traced, that is, the probable path of the sheet must be extended. The term "nappe" has been applied to the falling sheet; its application to the upper and lower surfaces of the sheet is also permissible. These may be called, then, the "upper" and "lower" nappes, for convenience.

The cases of vertical and inclined weirs depicted in Fig. 22 indicate the effect upon the upper and lower nappes of inclining a vertical, sharp crest down-stream. The weir shown in Figs. 19 and 20 should preferably conform to the lower nappe of the inclined weir, rather than to that of the vertical weir. On the other hand, high overfall dams usually have a vertical up-stream face, so, for this type, the nappe of the vertical weir of Fig. 22 should be employed to shape the crest. As the formulæ of design are general, with regard to the distance of the actual crest down-stream of the vertical up-stream face, and to the parabola of the down-stream face, any condition may be cared for by such proper considerations of the flow sheet in connection with determining the parabolic parameter for a given case.

As an aid to judgment, it may be stated that Bazin showed, from his experiments on sharp-crested weirs, both vertical and at various inclinations up- and down-stream:

(1) That the thickness $\left(\frac{e'}{b'}\right.$ of Fig. 21$\left.\right)$ of the nappe over the crest diminishes as the weir is inclined down-stream. This diminution, barely perceptible for weirs inclined up-stream, becomes much greater when we pass to weirs inclined down-stream.

(2) For a given inclination of weir, $\frac{e'}{b'}$ increases with the head, b', or rather with the ratio of b' to the height of the weir.

(3) Except for very low weirs, or weirs where the velocity of approach is more perceptible, the ratio of the height of the lower nappe to b' or the value of $\frac{b'-b}{b'}$ (see Fig. 21) appears to be independent of the head b' for a given inclination of weir. M. Boussinesq also stated $\frac{b'-b}{b'}$ to be a constant.

(4) The thickness of the sheet measured over the summit of the lower curve (at the crest line in Fig. 21) increases from the greatest observed inclination up-stream to that of 1 on 1, or 45°, down-stream, beyond which the thickness diminishes.

(5) For a weir of constant height, the ratio of the height of the lower nappe ($b'-b$, of Fig. 21) to b' diminishes as the inclination of the weir changes from up-stream to down-stream. The ratio of its chord length to b' (the chord being measured horizontally from the "theoretical" crest of Fig. 21) decreases as $\frac{b'-b}{b'}$ decreases.

(6) For weirs inclined up-stream, the inclination, although it modifies considerably the value of $\frac{b'-b}{b'}$, has but little effect upon that of $\frac{e'}{b'}$, which does not vary as much as 0.02 as we pass from a vertical weir to an up-stream inclination of 45°. For weirs inclined down-stream, however, $\frac{e'}{b'}$ diminishes rapidly, and the curve of the surface becomes elongated as the inclination increases.

Further experiments by Bazin on weirs of irregular section * indicated that, for different heads,† the ratio $\frac{e'}{b'}$ did not change very rapidly for the same weir, and that the ratio of the thickness, e', of the sheet of water at the up-stream edge of the sill, to the total head, b', varies within very extended limits for weirs with sloping faces. This variation is due to three causes, as follows:

(*a*) The "width" of the crest, or ratio of the head, b', to the crest width. For squared timbers, for example, the value $\frac{e'}{b'}$, quite large for small heads, diminishes as the head increases, approaching progressively to that which applies to the nappe of a sharp-crested weir ("in thin partition"). This influence ought naturally to be found again in weirs with sloping faces, the crest having a fixed width.

(*b*) Inclination of the down-stream face. When slightly inclined to the horizontal it exercises an influence similar

* "Annales des Ponts et Chaussées, Mémoires et Documents," 1898, 2me trimestre, pp. 121-264.

† Heads from 0.1 to 0.4 meter (3.9 to 15.7 inches).

to enlarging the crest and has the same effect in increasing the value of $\frac{e'}{b'}$.

(*c*) Inclination of the up-stream face. The effect of this inclination, which modifies the contraction of the sheet of water at the passage of the sill, is to diminish $\frac{e'}{b'}$.

These results for inclination of the faces * are analogous to those established for sharp-crested weirs; but it was observed that the values of $\frac{e'}{b'}$, instead of growing with the head, as for sharp-crested weirs, continue to decrease as for weirs of squared timber, thus fixing the influence of width of crest. From another point of view, it may be observed here that the effect of this virtual widening of the crest tends to increase the initial values of $\frac{e'}{b'}$, over the corresponding values for a weir of sharp crest, and these values of $\frac{e'}{b'}$ with increase of head, b', tend to approach the values of $\frac{e'}{b'}$ for a sharp-crested weir, hence a comparative diminishing of values of $\frac{e'}{b'}$ as the head is increased, may result.

From a further series of experiments, one set of which was upon a weir with a crest 15.7 inches wide and with up-stream face at a constant slope of 2 on 1, and a down-stream face inclination varying from 1 on 2, to 1 on 4 and 1 on 6, it was developed that the value of $\frac{e'}{b'}$ remained

* Crests of weirs varied from nearly 4 inches to nearly 8 inches in width.

nearly unchanged and its value ranged between 0.87 and 0.88.

Weirs with faces joined by arcs of circles to a horizontal crest, and above all weirs with curved profile, conduce to small values of $\frac{e'}{b'}$, but it should be remembered, in this connection, that the thickness e', measured properly at the "theoretical crest" or where the sill is entirely replaced by an edge, is not at all comparable with that observed at the most elevated point of the lower nappe. On weirs where the crest is joined to the up-stream face by a curved surface $\frac{e'}{b'}$ (presumably measured at the actual crest) reduces to below 0.80; but the crest is not then a thin edge.

Free nappes in "thin partition" give for $\frac{e'}{b'}$ values ranging from 0.85 to 0.86. The foregoing, considered in connection with Fig. 22, will, it is believed, indicate the effect upon the shape of the nappes of the inclination of the up-stream face, the breadth of crest, and the down-stream face slope near the crest.

Although objection may be raised against employing empirical data from observations of comparatively low heads for conditions that might obtain for high heads, it should be remarked that, as the head increased, both for the sharp-crested as well as for the irregularly shaped weirs, the variations of the relations approached better definition than for the low heads. Also, there should be noted the constant ratios that obtained in some of the phases outlined. In the absence of more extended data, therefore, the several indications suggested above may be employed.

It will suffice to consider, for example, the sharp-crested vertical weir, in connection with Figs. 22, 23, and 24. The problem remains to extend the nappes, as plotted by Bazin's co-ordinates, in terms of the head b'. In Fig. 23 verticals through the tenth marks, from $1'$ to $7'$, have been drawn.

The filament curve, *OPB*, of the average velocity is then traced, its position in the vertical (the crest line) through the actual crest being first ascertained.

An integration of each of the velocity curves of Fig. 24, between the upper and lower nappes and the vertical axis of co-ordinates to which it is referred, yields the discharge for each case (within about 5 per cent of the values obtained by the Francis weir formula, $Q = 3.33LH^{3/2}$). Each discharge divided by the vertical distances between the nappes at the section (Fig. 24) will give the value of the average velocity, whence its ordinate, or location above the actual crest, may be read from its velocity curve. This location was found to vary from 41 to 44 per cent of the water-sheet's thickness, above the actual crest. The average for the six heads observed was found to be 43 per cent.*

The value of the coefficient C in the formula for discharge, Q, given in Part I of this chapter, lies between 3.40 and 3.50. The average was found to be 3.48 in the cases given. H is taken upon the "theoretical" crest.

As has been brought out, the ratio, $\frac{b'-b}{b}$ (Fig. 21), is practically constant. The filaments passing the section

* Mr. Richard Muller, "Engineering Record," Oct. 24, 1908, uses $\frac{1}{3}$; but this value appears to involve pressure heads rather than velocities in their distribution through the sheet.

$X-X'$, Fig. 23, are sensibly parallel. Bazin demonstrated that the slight inclination in their direction did not affect the readings of the instruments obtaining the measurements of the velocities. Hence, we may proceed to lay off on each of the verticals $1-1'$, $2-2'$, etc., 0.43 $(1-1')$ 0.43 $(2-2')$, etc., above the curve of the lower nappe. The sheet section lines, of which $a-a'$ is one, are next drawn by making each pass through a point on a vertical, such as P, just located, and normal to the straight line joining the two similar points adjacent to P. On these lines, $a-a$, 0.43 of the distance of each, comprehended between the upper and lower nappes, is laid off above the lower nappe. These may be considered points on the path of the filament OPB of average velocity.

The discharge divided by the thickness of the sheet, such as at $a-a'$, will give the average velocity which may be plotted to scale as Vp in Fig. 23, normal to $a-a'$, and applied to the point of the filament curve nearest P. The vertical and horizontal components plotted, will, from their resulting values, enable the origin, O, of the filament curve to be located with sufficient precision. This may be tried by consideration of other points and an approximate location fixed. It will be found that the average velocity of the section $X-X'$ applies here, though the horizontal component V_h should be used for the sheet extension beyond section $a-a'$. With O and V_h fixed, the path OPB may be approximately computed with reference to O and plotted, by choosing points down the curve and computing the vertical component of the velocity attained, by the laws of falling bodies. Laying off this vertical component for each point, and plotting V_h, results

in obtaining the velocities of the sheet, tangential to *OPB*, for the various points. The discharge divided by the tangential velocity at each point gives the thickness of the sheet. This, laid off on a line normal to the tangential velocity and through the given point, with 0.43 below the point, and 0.57 of the thickness of the sheet above the point, determines two points, the one on the lower nappe, and the other on the upper nappe. This procedure may be continued as far down as desirable and smooth curves drawn through the points, so determined, will give the direction and shape of the sheet with sufficient accuracy.

A parabola whose parameter is about $1.8b'$, or $K = 1.8$, will approximate the position of the lower nappe. Its origin is at the highest point of the lower nappe, in the crest line, therefore, for a cross-section to be well within the sheet, its K should be greater than $K = 1.8$. $K = 2.25$ will provide a parabola that will meet this requirement adequately and allow the curve of the face between the crest and $7'$ (Fig. 23) to be somewhat flattened by a curve of longer radius, as suggested earlier.

A parabolic curve for the nappes of the inclined weir shown in Fig. 22 could be similarly worked out.

It is reasonable to suppose that the friction of an actual masonry surface upon which the lower nappe is flowing would modify the thickness of the sheet. This would be probable, from a consideration of the similarity in effect of the crest of masonry here and the broad crest effects on the thickness, cited earlier from experiments; however, the curves of the face would tend to have an opposite effect from that caused by the virtual widening

of the crest; but to what extent cannot be said at present. It is possible that the horizontal component of the velocity near the crest would be cut down in value from that obtaining with discharge free, and over a sharp crest; but, in basing the determination of a cross-section upon the flow over a sharp crest, there is obtained a larger trajectory for the falling sheet than might otherwise result, and so a preferable cross-section will result, as it is on the safe side, both as to encroachment within the water sheet and stability.

CHAPTER VII

THE ARCH DAM

BEFORE deciding upon the cross-section for a masonry dam, the proposed site should be carefully studied with regard to its topography, to determine the type of structure that can be most advantageously used with existing local conditions, keeping in mind especially the question of economy of material.

It is evident that under all circumstances the choice must lie between the gravity type of dam heretofore discussed and any one of the arched types about to be briefly touched upon. But where gorges or canyons are encountered, the selection of the arch most naturally suggests itself, especially since in those of 200 to 500 feet in width moderate spans result.

The economy of one form over another will depend upon whether, with the greater length but smaller cross-sectional area, the arched type will require more or less material than the straight gravity type with its shorter length but greater cross-sectional area.

Should the gravity section be discarded in favor of the other form, the further question arises as to which of the various arched types may be used to best advantage, and it is the purpose of the few following paragraphs to refer briefly to these types before proceeding to the discussion of the design of what may be termed a simple arched dam.

Buttress Arch Type. — This form of dam consists of a series of plain or reinforced concrete arches, either vertical or inclined down-stream, supported at the abutments by buttresses, and may need special attention, particularly if the length of the proposed structure is considerable, to determine whether it may be economically used in a given location. Such an investigation may be made by first considering the arches in connection with the loading producing the stresses in them, and second by analyzing the buttresses with respect to the component arch thrusts transmitted to them and acting in a down-stream direction.

The former investigation would be undertaken by means of any of the prevailing arch theories applicable to the case in hand, while the latter would be prosecuted by the use of the formulæ already established in connection with the gravity type of dam. To apply these formulæ it is necessary to consider a unit of thickness of the buttress and to reduce the component down-stream thrust of two adjacent arches acting upon a single buttress to an equivalent hydrostatic pressure, so that the conditions may correspond to those in the design of the gravity type dam, where the water pressure is taken as acting over a unit length of the dam.

This reduction may be accomplished by calculating an equivalent value of γ, the weight of a cubic foot of water, whereupon with the thickness of the buttress assumed for each level or stage, and with the masonry density either assumed or known, the formulæ for design may be employed directly to solve for the successive lengths of base, l, of the buttress. The sides of the buttresses at each stage may be assumed, for ease, to be vertical planes,

until the calculations are completed, when a proper batter may be given to avoid the successive offsets which would result. It may be advisable also to investigate the buttresses as "plates" to determine their probable tendency to "buckle." One of the chief advantages of this type of dam, as is quite evident, is its comparative freedom from the effect of uplift.

Gravity Section.—A gravity section may be arched in plan, or a special cross-section for the arch type may be developed, since the arch action may, except for high stresses, be limited in the thicker cross-section. In the gravity sections arched in plan, it may be shown that the arch takes from 5 to 8 per cent of the load, or expressed differently, that the cantilever transmits most of the load to the foundation.

Arch Section.—The section of an arch dam shown in Fig. 25 was investigated by the method described later, and the study, including other dams of like dimension, except for down-stream face batters and bases, led to the conclusion that by thickening the top to at least 5 or 6 feet, thus gradually increasing the thickness until it reached the section of Fig. 25 at a depth of about 80 feet, a section could be obtained which would reduce the arch stresses at the top without increasing them below at all appreciably.

The above-mentioned figure represents in a general way the cross-section of an arched dam, and by the curves indicates the amount of arch action at the various levels. It is evident from the curves that at the foundation no arch action exists. The up-stream radius was taken as 350 feet, and the arch span as 600 feet.

To obtain a tentative cross-section for an arch dam,

the thin ring formula may be used to calculate the bases at the different levels.

$$l = \frac{PR_n}{q}.$$

l = length of base in feet;

P = water pressure in pounds per square foot;

R_n = length of the up-stream radius in feet;

q = average stress in the ring in pounds per square foot.

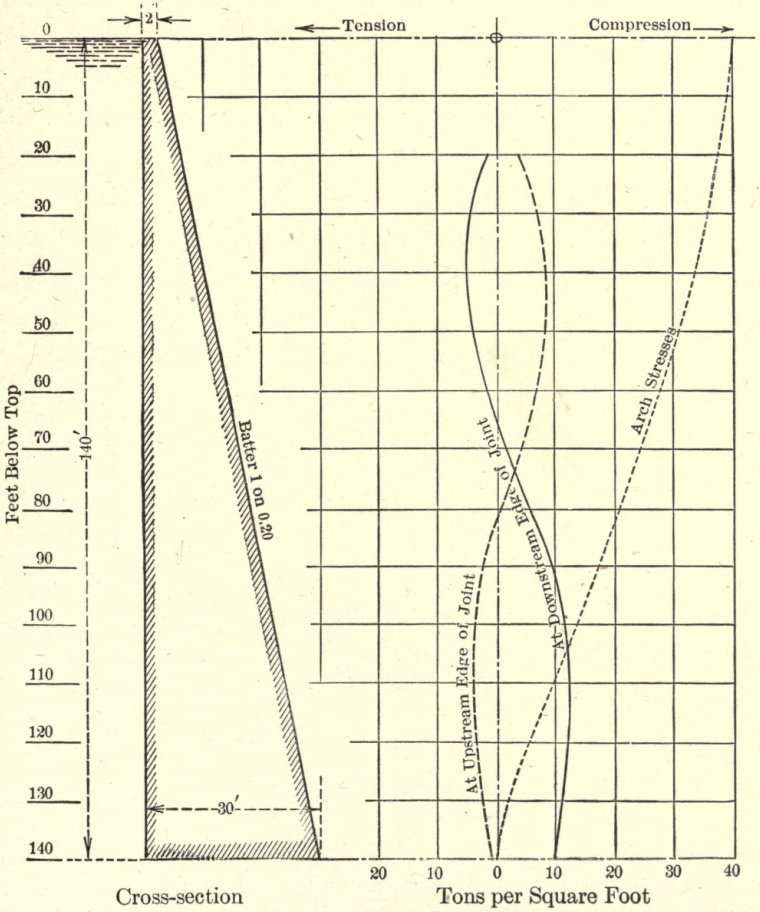


FIG. 25.

The Constant Angle Dam.—It may be shown that to obtain an arch dam of minimum volume, when the structure acts as an arch, and with minimum stresses (even near the foundation), any arch slice must be subtended by a horizontal central angle, between abutments, of 133°–34′.* Practically, this angle may be reduced to 120°. The fact that this type of dam has an ability to act as an arch, to a much greater degree than the ordinary arch dam, follows from the fact that an arch, when loaded, undergoes a deflection proportional to the square of the up-stream radius (see Eq. 8, page 149), and in the "constant angle" type, this radius may be several times shorter at the foundation than at the top. As a consequence, the deflection at the base required for the same unit stress would be proportionately less than the deflection required at the top to produce this same unit stress, and as canyons are generally narrower at the bottom than at the top, this condition usually applies.

The principle underlying the "constant angle" may be developed as follows:

From the formula just given:

$$l = \frac{PR_n}{q},$$

it is evident that the base l and therefore the cross-sectional area, varies directly as the radius R_n. The volume in a given section, however, is equal to the area times the length of the mean arc, which latter may be expressed in terms of the length of the mean radius (designated as R_m)

* "The Constant Angle Arch Dam," by Mr. Lars R. Jorgensen. Trans. Am. Soc. C.E., Vol. 78, p. 685.

times the subtended angle (designated as $2\phi_n$, expressed in terms of π). We may therefore write for the volume V:

$$V = \text{Area} \times R_m \times 2\phi_n.$$

But R_m, the mean radius, is equal to one-half the span, or width of canyon W, divided by $\sin \phi_n$, or, expressed as an equation:

$$R_m = \frac{\frac{1}{2}W}{\sin \phi_n}.$$

Since the area of the cross-section is proportional to the length of the radius, R_n, and the volume to R_m, we may write without sensible error:

$$V = C \times \frac{(\frac{1}{2}W)^2 \times 2\phi_n}{\sin^2 \phi_n} = \frac{K\phi_n}{\sin^2 \phi_n}.$$

This expression, in which C and K are constants, K depending upon the width of the canyon, indicates that the volume varies with the term $\frac{\phi_n}{\sin^2 \phi_n}$.

If we differentiate this expression with respect to ϕ_n, and equate the differential coefficient to zero for a minimum, we have,

$$\phi_n = \frac{1}{2 \cot \phi_n},$$

which equation is satisfied by the value of $66°-47'$ for ϕ_n, whence $2\phi_n$ equals $133°-34'$.

Poisson's ratio for lateral strains is taken into consideration in determining the relative arch action in a constant angle type, a value of one-fifth being adopted, and the initial stresses induced axially by the weight of

the dam on the foundation, together with the water load, being utilized to help support the latter.

Fig. 26 represents a cross-section of a dam, developed by the constant-angle principle, 250 feet high, with a base of 70 feet and a cross-sectional area of 9,668 square feet. An equivalent rectangle of equal base would have to be 138 feet high.

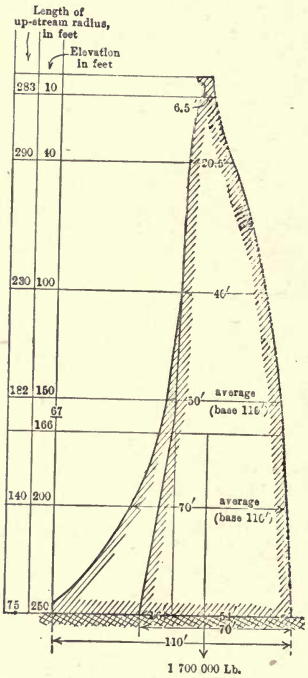


FIG. 26.

If the specific gravity of the concrete be assumed at 2.3, the average vertical pressure may be expressed as $\frac{2.3H}{\alpha}$, where H is the height of the dam, and α is the ratio of the total height to that of the equivalent rectangle, whence

$$\alpha = \frac{250}{138} = 1.81.$$

The mean vertical compression for this section would then be $\frac{2.3H}{1.81} = 1.27H$ with the reservoir empty, expressed in terms of the head of water, when the latter is at an elevation of the top of the dam.

With the reservoir full, the radial water pressure is assumed to counteract the strain of the masonry acting in an up-stream and down-stream direction, although there is, of course, no direct opposing force on the down-stream side, acting horizontally up-stream.

It is reasonable to assume, however, that the reactions

at the abutments at least partially provide an indirect force acting from the down-stream side in an up-stream direction.

As Poisson's ratio is approximate, the full head is used as active in this respect, hence the total resulting initial axial compression at the foundation for the section may be written as

$$\tfrac{1}{5}(1.27H+H)=0.45\times H.$$

The height of water, represented by h, that this initial axial compression of $0.45H$ will resist without causing the arch at the bottom to shorten in length, will result by use of the ring formula

$$l=\frac{PR_n}{q},$$

in which q is represented by $0.45H$ and P by h, whence

$$h=\frac{l}{R_n}\times 0.45H.$$

For the narrow section of Fig. 26,

$$R_n=75', \quad l=70',$$

hence this section, by application of the above expression, is able to carry as an arch

$$h=0.45H\times\frac{70}{75}=0.42H,$$

or 42 per cent of the total head of water, before any shortening in the length of the arch occurs. The remaining 58 per cent of loading is distributed between the arch and cantilever as explained later on.

Arched Dam Investigation.—After the plan and cross-section of an arched dam have been settled upon, the structure may be investigated to determine the proportion of the loading resulting from horizontal water pressure which will be cared for by the structure acting as a horizontal arch, and that cared for by the structure acting as a vertical cantilever, respectively. From this resulting distribution of loading, the value of the intensities of stress on vertical planes normal to the axis of any arch ring under consideration may be calculated from the resulting thrusts into the sides of the canyon. Stress intensities for horizontal joints may be found by combining the stresses due to the horizontal loads assumed by the vertical cantilever, acting within the elastic limit of the material, with the stresses due to the weight of the dam above the joint in question.

This investigation for distribution of loading between arch and cantilever may be made independently of the value of the modulus of elasticity of the concrete or material of which the dam is constructed, if considered homogeneous, as will appear.

Limitations.—If, however, the dam be built in sections with transverse, vertical "contraction" joints, dividing it segmentally into portions approaching voussoirs in their nature, it could, under certain conditions, hardly be considered to act as an elastic arch. These joints may be open more or less at times, according to the atmospheric temperature, the season of the year when masonry between them was laid, and the depth of water behind the dam, with its consequent effects of swelling the masonry and affecting its internal temperature. Furthermore, the con-

traction at any time may be greater at the top than further down, or within the dam and different segments may be simultaneously in different conditions of stress due to contraction.

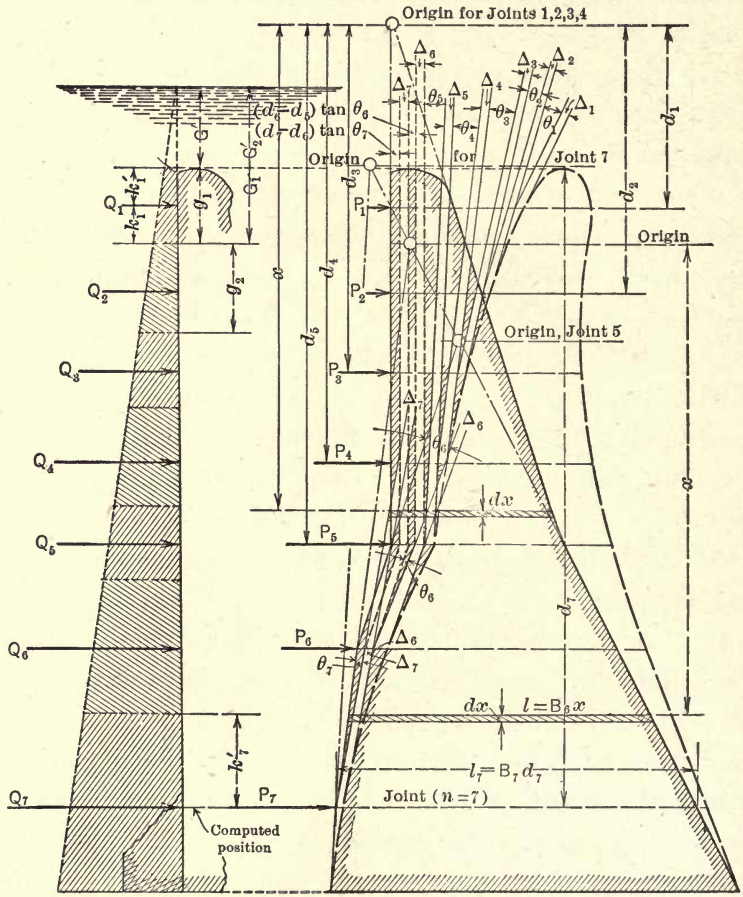
In short, the conception of the dam as a horizontal arch, fixed at the ends and along the foundation of the dam and acting to a greater or less degree as a huge voussoir arch, involves both a consideration of internal temperature conditions and a knowledge of the value of the modulus of elasticity of the great mass of the dam, both of which are sufficiently uncertain to render anything but an extended investigation under various assumptions of doubtful value.

But the limits between which the behavior of the structure may lie, viz., that of an elastic arch, held at the sides and bottom of the gorge, and that of a cantilever, bearing the total load, may therefore be profitably investigated.

Again, the deflection of such cantilever with an assumed value for the modulus of elasticity of the material may be considered, together with the deflection of the topmost arch slice at the crown, due to the opening of the contraction joints. These may be compared, or they may be reduced to a corresponding temperature range and compared with the maximum possible range at the site of the dam, whence an indication as to the probability of arch action ceasing wholly or in part may be reached.

Method of Arch and Cantilever Analysis.—For the consideration of the elastic arch with cantilever action and no contraction joints, or joints tightly closed, the following method is elaborated from a method outlined by

the late R. Shirreffs in a discussion of a paper on the Lake Cheesman Dam and Reservoir by the late Charles L.



EXAMPLE:

(1) For battered up-stream face,

$$D_2=\Delta_2+\Delta_3+\Delta_4+\Delta_5+\Delta_6+\Delta_7+(d_3-d_2)\tan\theta_3+(d_4-d_2)\tan\theta_4+(d_5-d_2)\tan\theta_5+(d_6-d_5)\tan\theta_6+(d_7-d_6)\tan\theta_7.$$

(2) For vertical up-stream face (assume Joint 5 to be base of dam, $\theta_5=0°$), then,

$$D_2=\Delta_2+\Delta_3+\Delta_4+(d_3-d_2)\tan\theta_3+(d_4-d_2)\tan\theta_4.$$

Note: All joints may be referred to any one origin.

FIG. 27.

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The formulæ developed in the following pages are applicable to the vertical cantilever, contained between two vertical, radial planes (1 foot apart at the extrados), of any arched dam, either of overfall (spillway) or complete retaining type.

The down-stream face, though it may be curved in vertical profile, should be considered straight between load-points. Thus, in a spillway dam, the load-points at the more curved portions of the vertical poofile may be taken nearer together. The load-points may be arbitrarily chosen in positions just so that the portions between successive load-points may be considered as trapezoids without essentially altering the cross-section of the dam. The "joints" are taken at the load-points. Generality has been attained by introducing the expression for the moment of inertia of the horizontal cross-section of this vertical cantilever in terms of the variable, x, before integrating.

The following nomenclature, together with that shown in Fig. 27, applies.

Nomenclature

For the Cantilever.—The "origin" of a joint is the point of intersection of the down-stream side, or face, of the dam, next below that joint, with the corresponding up-stream face of the dam, both produced, if necessary.

B = the batter of the down-stream side (or, if the up-stream side is battered also, the combined batter

* Trans. Am. Soc. C. E., Vol. 53, p. 155.

of the down-stream and corresponding up-stream side) next below the joint, or load-point, in question.

x = the vertical distance from the origin of any joint of the cantilever to any level in the portion immediately below the joint (between it and the next joint).

d_n = the vertical distance of the joint n below its origin (d signifies general expression).

$l = Bx$ = the length of a horizontal joint of masonry at the depth x below the origin for that joint (same as thickness of arch ring).

l_n = the length of a horizontal joint of masonry at the depth d_n.

l_{n+1} = the length of a horizontal joint of masonry at the depth d_n+1, etc.

m = the total number of load-points, or joints.

n = the number of the load-point, or joint, considered, beginning with the top load-point.

Q_n = the total load of water, in tons, considered as concentrated at any joint, n, over 1 foot length of extrados. (See left-hand diagram, Fig. 27.)

P_n = that part of Q_n assumed by cantilever action at the center of the dam.

E = modulus of elasticity of the material of the dam.

I = the moment of inertia of the horizontal cross-section of the cantilever, at the level, x. (See Appendix I for derivation.)

$$I = \frac{6R_n^2B^3x^3 - 6R_nB^4x^4 + B^5x^5}{36R_n(2R_n - Bx)}, \quad . \quad . \quad . \quad (1)$$

where R_n is the radius of the arch extrados at the portion of the cantilever to which x is taken.

M_n = the moment of all loads above the joint considered (joint n), (M signifies such moment in general).

Δ_n = the deflection of each individual portion of the cantilever produced by all of the loads above that portion (see Fig. 27), (Δ signifies such deflection, in general).

θ_n = the angle of deflection (change of angle) at any load-point, n, having reference only to the portion of the cantilever between that load-point and the one next below $(n+1)$. (See Fig. 27). (θ signifies such angle in general.)

D_n = the total deflection of the cantilever at any load point, n.

For example, from Fig. 27,

$$D_2 = \Delta_2 + \Delta_3 + \Delta_4 + (d_3 - d_2) \tan \theta_3 + (d_4 - d_2) \tan \theta_4$$

for a dam whose up-stream face is vertical. (See p. 144.)

g_n = the vertical extent of hydrostatic pressure, the resultant of which

(Q_n) is concentrated at the joint considered, or load-point n.

$$\left.\begin{array}{l} k'_n = \text{the portion of } g_n \text{ above joint } n \\ k_n = \text{the portion of } g_n \text{ below joint } n \end{array}\right\} \text{ or } g_n = k_n + k'_n. \quad (2)$$

G'_n = the head of water on the level at the upper end of the distance g_n.

G_n = the head of water on the level at the lower end of the distance g_n. (For example, $G_1 = G'_2$.)

The following two expressions (readily derived) give the relations among g_n, k_n, k'_n, G'_n, and G_n:

$$g_n = \sqrt{3k'_n G'_n + [\tfrac{3}{4}(G'_n - k'_n)]^2} - \tfrac{3}{4}(G'_n - k'_n), \quad . \quad . \quad (3)$$

$$g_n = \sqrt{-3k_n G_n + [\tfrac{3}{4}(G_n + k_n)]^2} + \tfrac{3}{4}(G_n + k_n), \quad . \quad . \quad (4)$$

also,

$$k'_n = g_n \frac{G'_n + 2G_n}{3(G'_n + G_n)}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4a)$$

and are convenient for computing the hydrostatic concentrations for load-points wherever chosen. (See Fig. 27.)

Eq. (4*a*) may be used to locate the last concentration at the lower part of the dam.

Q_n, in tons, may be computed from Eq. (5), following:

$$Q_n = \frac{g_n}{32}\left(G'_n + \frac{g_n}{2}\right), \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

if G'_n and g_n are expressed in feet.

For the Arch.—(Arch ring depth taken at 1 foot.)

R_n = the radius of extrados at load-point n. (If radius of extrados is constant, throughout, $R_n = R$.) In equations, the extrados radius will be designated by R_n.

r_n = the radius of the axis of the arch at any load-point, n.

$$r_n = R_n - \frac{l_n}{2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

q_n = unit loading on axis of arch (corresponding to extrados unit loading) at load-point n.

$$q_n = \frac{R_n}{r_n}\left(\frac{Q_n - P_n}{g_n}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

$\phi_n = \frac{1}{2}$ central angle of arch span at level of load-point, n.

$l = Bx$ = thickness of arch ring (depth 1 foot) at any level of dam.

l_n = same at level d_n = length of joint n.

D_c = deflection at crown of arch at level of load-point (cantilever "joint.") n. (See Appendix II.)

$$D_c = \left\{ \frac{\frac{2 \sin \phi_n}{\phi_n}(1 - \cos \phi_n) + \cos^2 \phi_n - 1}{\frac{3 \phi_n}{\sin \phi_n} + \cos \phi_n - 4} + (1 - \cos \phi_n) \right\} \frac{q_n r_n^2}{E l_n}. \quad (8)$$

$$CC_c = \frac{\frac{2 \sin \phi_n}{\phi_n}(1 - \cos \phi_n) + \cos^2 \phi_n - 1}{\frac{3 \phi_n}{\sin \phi_n} + \cos \phi_n - 4} + 1 - \cos \phi_n$$

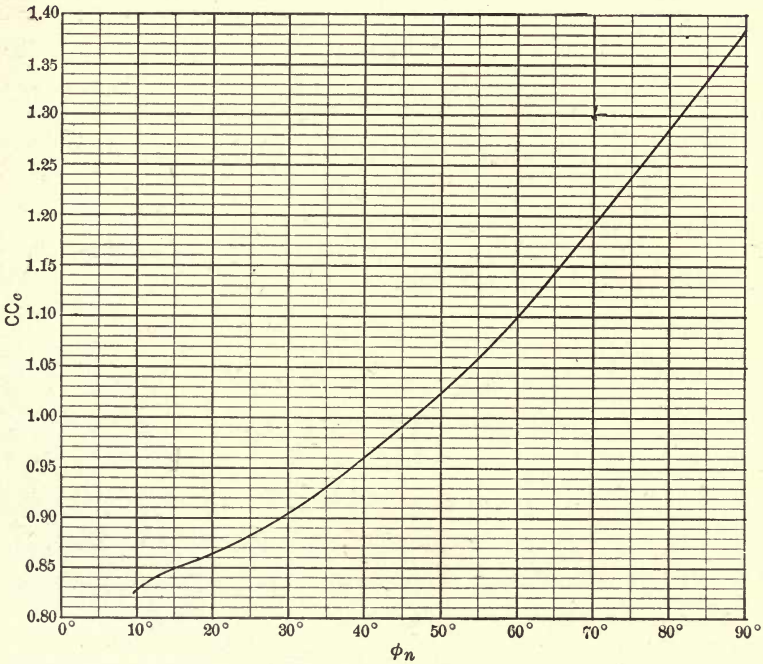

FIG. 28.

The trigonometric function of ϕ_n, in Eq. (8), may be denoted by CC_C and can be plotted as a curve, greatly simplifying its application, in the calculations. (See CC_c of Fig. 28.)

The Cantilever. — The general differential expressions for flexure of a cantilever may be written

$$d\Delta = \frac{M(x-d)dx}{EI} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

$$d\theta = \frac{Mdx}{EI}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

in which there may be substituted

$$M_n = P_1(x - d_1) + P_2(x - d_2) + \ . \ . \ . \ + P_n(x - d),$$

which may be transformed to

$$M_n = (P_1 + P_2 + P_3 + \ . \ . \ . \ + P_n)x$$
$$- (P_1 d_1 + P_2 d_2 + P_3 d_3 + \ . \ . \ . \ + P_n d_n). \quad (11)$$

Also, there may be written, for an arch dam with vertical up-stream face

$$D_n = (\Delta_n + \Delta_{n+1} + \Delta_{n+2} + \ . \ . \ . \ + \Delta_m) + (d_{n+1} - d_n) \tan \theta_{n+1}$$
$$+ (d_{n+2} - d_n) \tan \theta_{n+2} + (d_{n+3} - d_n) \tan \theta_{n+3} + \ . \ . \ .$$
$$+ (d_m - d_n) \tan \theta_m. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

Where the up-stream face is battered, as shown below joint No. 5 in Fig. 27, let m' be the number of the joint at the base of the vertical portion, or joint 5, as there shown. Eq. (12), with this special significance applied in this case to m', will hold as written, down as far as the base of the portion with vertical up-stream face, if n_5 be changed to m'.

The additional terms, made up of differences of d's times respective tangents of θ's, will be successively of the form,

$$(d_{m'+1} - d_{m'}) \tan \theta_{m'+1} + (d_{m'+2} - d_{m'+1}) \tan \theta_{m'+2} + \ . \ . \ .$$
$$+ (d_m - d_{m-1}) \tan \theta_m;$$

or, for a dam with *part vertical* and *part battered up-stream* face, as shown in Fig. 27,

$$D_n = (\Delta_n + \Delta_{n+1} + \Delta_{n+2} + \ldots + \Delta_m) + (d_{n+1} - d_n) \tan \theta_{n+1}$$
$$+ (d_{n+2} - d_n) \tan \theta_{n+2} + (d_{n+3} - d_n) \tan \theta_{n+3} + \ldots$$
$$+ (d_{m'} - d_n) \tan \theta_{m'} + (d_{m'+1} - d_{m'}) \tan \theta_{m'+1}$$
$$+ (d_{m'+2} - d_{m'+1}) \tan \theta_{m'+2} + \ldots + (d_m - d_{m-1}) \tan \theta_m. \quad (12a)$$

In case there is *no vertical up-stream face* (see lower portion of right-hand diagram, Fig. 27) Eq. (12*a*) would take the following form:

$$D_n = (\Delta_n + \Delta_{n+1} + \Delta_{n+2} + \ldots + \Delta_m) + (d_{n+1} - d_n) \tan \theta_{n+1}$$
$$+ (d_{n+2} - d_{n+1}) \tan_{n+2} + \ldots + (d_m - d_{m-1}) \tan \theta_m. \quad (12b)$$

Tan θ, in Eqs. (12), (12*a*) and (12*b*) above, may be considered equal to θ, in each case.

Cantilever Deflection Equations.—By substituting in Eqs. (9) and (10), above, the expressions for I and M, given in Eqs. (1) and (11), respectively, and integrating with respect to x, the resulting expressions between the limits $x = d_{n+1}$ and $x = d_n$, and reducing, there may be obtained the respective expressions for Δ_n and θ_n (Eqs. (13) and (14)). (See Appendix I.)

By means of Eqs. (13) and (14), p. 153, the Δ and θ for each joint or load-point can be computed, and combinations, as indicated by any one of Eqs. (12), (12*a*), or (12*b*) that pertains, made for each joint, whence there results for each such joint an equation involving E, and the various P's (the latter being the unknowns) for the deflection, D_n.

Arch Deflection Equations.—Eq. (8), just preceding, provides for writing the several expressions for the crown deflections of the arch elements of the dam, one expression for each of the load-points assumed. These, too, will be in terms of E and the various P's, as an inspection of Eq. (7) in connection with Eq. (8) will show.

Resulting Simultaneous Equations. — By equating each expression for the arch crown deflection, to the expression for the cantilever deflection at the same level, or load concentration-point, a series of simultaneous equations may be evolved, E dividing out each time, and P_1 to P_m appearing in each of the m equations as the unknown quantities.

Solution for Distribution of Loading between Arch and Cantilever Actions.—A solution of this set of simultaneous equations will yield the value of the P for each joint or load-point. These loads assumed by the cantilever action may then be subtracted from the respective total hydrostatic loadings (Q) whence the amount assumed by each of the horizontal arch laminæ results.

The fact is neglected, however, that the several arch slices throughout the dam actually cannot move freely, in relation to each other. This would tend to stiffen the dam along the arch axis and thereby transmit the arch thrusts in an axial direction from a given arch into the abutment of some lower arch slice.

The general expressions for $E\Delta_n$ and $E\theta_n$ follow as Eqs. (13) and (14). These, with equations for I and D_c, (Eqs. (1) and (8)), are derived, as outlined above, in Appendices I and II.

$$E\Delta_n = \frac{(P_1d_1 + P_2d_2 + \ldots + P_nd_n)}{B^2}\left\{\frac{2(2l_n - 3R_n)}{R_n^2}\right.$$

$$\times 2.30259 \log \frac{d_{n+1}}{d_n} + \frac{6(l_n - 2R_n)(d_{n+1} - d_n)}{R_n l_n d_{n+1}} + \frac{6(d^2_{n+1} - d_n^2)}{l_n d^2_{n+1}}$$

$$- \frac{(2-\sqrt{3})l_n - (3-\sqrt{3})R_n}{R_n^2} \times 2.30259 \log \frac{l_{n+1} - (3+\sqrt{3})R_n}{l_n - (3+\sqrt{3})R_n}$$

$$\left. - \frac{(2+\sqrt{3})l_n - (3+\sqrt{3})R_n}{R_n^2} \times 2.30259 \log \frac{l_{n+1} - (3-\sqrt{3})R_n}{l_n - (3-\sqrt{3})R_n}\right\}$$

$$+ \frac{(P_1 + P_2 + \ldots + P_n)}{B^3}\left\{\frac{12R_n - 6l_n}{R_n} \times 2.30259 \log \frac{d_{n+1}}{d_n}\right.$$

$$- \frac{12(d_{n+1} - d_n)}{d_{n+1}} + \frac{(3-\sqrt{3})l_n - 6R_n}{R_n}$$

$$\times 2.30259 \log \frac{l_{n+1} - (3+\sqrt{3})R_n}{l_n - (3+\sqrt{3})R_n} + \frac{(3+\sqrt{3})l_n - 6R_n}{R_n}$$

$$\left.\times 2.30259 \log \frac{l_{n+1} - (3-\sqrt{3})R_n}{l_n - (3-\sqrt{3})R_n}\right\} \quad \ldots\ldots\ldots \quad (13)$$

$$E\theta_n = \frac{(P_1d_1 + P_2d_2 + \ldots + P_nd_n)}{B}\left\{-\frac{4}{R_n^2} \times 2.30259 \log \frac{d_{n+1}}{d_n}\right.$$

$$- \frac{6(d_{n+1} - d_n)}{R_n l_n d_{n+1}} - \frac{6(d^2_{n+1} - d_n^2)}{l_n^2 d^2_{n+1}} + \frac{2-\sqrt{3}}{R_n^2}$$

$$\times 2.30259 \log \frac{l_{n+1} - (3+\sqrt{3})R_n}{l_n - (3+\sqrt{3})R_n} + \frac{2+\sqrt{3}}{R_n^2}$$

$$\left.\times 2.30259 \log \frac{l_{n+1} - (3-\sqrt{3})R_n}{l_n - (3-\sqrt{3})R_n}\right\}$$

$$+ \frac{(P_1 + P_2 + \ldots + P_n)}{B^2}\left\{\frac{6}{R_n} \times 2.30259 \log \frac{d_{n+1}}{d_n}\right.$$

$$+ \frac{12(d_{n+1} - d_n)}{l_n d_{n+1}} - \frac{3-\sqrt{3}}{R_n} \times 2.30259 \log \frac{l_{n+1} - (3+\sqrt{3})R_n}{l_n - (3+\sqrt{3})R_n}$$

$$\left. - \frac{3+\sqrt{3}}{R_n} \times 2.30259 \log \frac{l_{n+1} - (3-\sqrt{3})R_n}{l_n - (3-\sqrt{3})R_n}\right\} \quad \ldots \quad (14)$$

These expressions involve only P, d, B, R, l, and E.

In use of expressions (13) and (14) above, for any joint, n, to which a particular batter, B, naturally applies (with reference to that joint's particular origin), care should be taken that the various d's (cf. P_1d_1, P_2d_2, above) are referred to that particular origin of the given joint n.

It will usually be found that, in Eq. (14) (for $E\theta_n$), two terms, the fourth in the first pair and the third in the last pair of braces are negligible.

Application of Foregoing Formulæ (13), (14) and (12), (12a), or (12b), (cantilever equations). — The procedure will be conveniently described by referring to forms, in order, for tabulating the factors that enter into the construction of the cantilever equations of deflection; that is, the application of the foregoing formulæ (13), (14), and (12), (12*a*), or (12*b*). Numerical values are given for illustration only, and to facilitate comprehension of the use of the tabulation forms.

The first step after the cross-section has been fixed, is to choose the load-points, or joints. In an overfall dam these will be more numerous at the upper and lower portions, where the curvature of the down-stream face necessitates shorter tangents to approximate more nearly the curved cross-section by one of rectilinear sides, forming a series of trapezoids, with the joints. The load-points may number from 4 to 10, according to the type of dam and its height and shape.

Second, calculate the g's by means of Eqs. (3) or (4) and for the last joint, m, locate the concentration for the distance g_m, remaining, by means of Eq. (4*a*). (See Fig. 27.)

Third, from the foregoing, compute by Eq. (5) the

Q's. The heads on the load-points may be entered into the tabulation as well, together with the lengths of the joints (l's).

These may be tabulated conveniently as in Table V.

TABLE V

$n=$	G'_n	k'_n	g_n	$G_n+\frac{g_n}{2}$	$\frac{g_n}{32}$	Q_n (tons)	Head on Q_n	l_n
1			*					
2			*					
3			*					
.								
.								
.								
.								
.								
m		*						

* Computed.

The fourth step is to compute the origins of the joints (see nomenclature for definition of origin). These, once found, should be placed as shown in Table VI, entitled "Distances of Origins above Joints," together with these distances, which are useful in carrying out the provisions of origin reference, noted immediately after and referring to use of Eqs. (13) and (14).

Suppose the position is desired of, say, Joint No. 2 with reference to the origin of Joint No. 3. In the left-hand column of Table VI, No. 3 gives the line and 2 the column, and in line 3 and column 2 is read 44.63. That is, the origin of Joint No. 3 is 44.63 feet *above* joint No. 2, or No. 2 is 44.63 feet *below* the origin of Joint No. 3. Where these distances are negative, the distance d in the formulæ must be written with the minus sign.

TABLE VI

Distances of Origins above Joints (in feet).

Joint No.	1	2	3	. . .		8	9	Base.
Origin of 1	12.18	21.73						9.55
Origin of 2	20.26	29.81	37.62					17.36
Origin of 3	35.08	44.63	52.45	64.26				29.18
Origin of ⋮								
Origin of ⋮								
Origin of 8	−41.42	−31.87	−24.05			65.34	75.34	116.76
Origin of 9	−79.98	−70.43	−62.61			26.78	36.78	21.76

TABLE VII

Coefficients for θ, in Equations for D

(Table of Differences)

	d_1	d_2	d_3	d_4		d_8	d_9
d_{n+1}, d_{n+2}, etc d_2	9.55				. . .		
d_3	17.37	7.82					
d_4	29.19	19.64	11.82				
⋮							
⋮							
d_8	106.76	97.21	89.39	77.57			
d_9	116.76	107.21	99.39	87.57	. . .	10.00	
Base	121.76	112.21	104.39	92.57	. . .	15.00	5.0

At this step in the work, the coefficients for θ, in cantilever equations for D, might be compiled, as indicated in Table VII.

For instance, in computing D_2, it is necessary to use as a factor, among others, $d_4 - d_2$. From Table VII, as given there, for example, the value 19.64 feet corresponds.

What is really the fifth step is to compute the factors occurring in Eqs. (13) and (14) as tabulated below, in Table VIII.

Each large factor is here designated by a Roman numeral, for future reference.

TABLE VIII

Joint No.	(I) $\frac{2.30259}{R_n} \log \frac{d_{n+1}}{d_n}$	(II) $\frac{6(d_{n+1}-d_n)}{l_n R_n d_{n+1}}$	(III) $\frac{12(d_{n+1}-d_n)}{d_{n+1}}$	(IV) $\frac{6(d^2{}_{n+1}-d_n{}^2)}{l_n d^2{}_{n+1}}$
1	0.0013898	0.0027862	etc.	etc.
2	0.0015531	0.0018624	5.032262	1.0303393
3	etc.	etc.	etc.	etc.
etc.				

Joint No.	(V) $\frac{2.30259}{R_n} \log \frac{l_{n+1}-(3+\sqrt{3})R_n}{l_n-(3+\sqrt{3})R_n}$	(VI) $\frac{2.30259}{R_n} \log \frac{l_{n+1}-(3-\sqrt{3})R_n}{l_n-(3-\sqrt{3})R_n}$	B	B^2	B^3
1	etc.	etc.	etc.	etc.	etc.
2	−0.00000481	−0.0000181	0.09285	0.00862	0.000800
3	etc.	etc.	etc.	etc.	etc.
etc.					

Also tabulate for reference, the values of the following:

R_n, R_n^2,

$\sqrt{3} = 1.732$ $\quad \frac{2-\sqrt{3}}{R_n^2}$; $\frac{2-\sqrt{3}}{R_n}$. $\frac{3-\sqrt{3}}{R_n}$; $(3-\sqrt{3})R_n$.

$2-\sqrt{3} = 0.268$

$2+\sqrt{3} = 3.732$ $\quad \frac{2+\sqrt{3}}{R_n^2}$; $\frac{2+\sqrt{3}}{R_n}$; $\frac{3+\sqrt{3}}{R_n}$; $(3+\sqrt{3})R_n$.

$3-\sqrt{3} = 1.268$

$3+\sqrt{3} = 4.732$

Tabulate for the various joints $\frac{1}{B}$, $\frac{1}{B^2}$, and $\frac{1}{B^3}$, the first two of these three quantities for calculating the θ's, and the last two for the Δ's. These could be added in three extra columns to Table VIII. $\frac{1}{B}$ and $\frac{1}{B^2}$ could be multiplied by the various d's and tabulated, to facilitate calculations of $E\Delta$'s below.

Eqs. (13) and (14) may now be written in terms of (I), (II), (III), etc., of Table VIII, with substitutions for l_n and R_n; that is, if the portions within the braces of Eqs. (13) and (14), involving (I), (II), (III), (IV), etc., be designated by S_n and S'_n and S''_n and S'''_n, there may be written:

$$E\Delta_1 = \frac{1}{B_1^2}P_1d_1S'_1 + \frac{1}{B_1^3}P_1S_1,$$

$$E\Delta_2 = \frac{1}{B_2^2}(P_1d_1 + P_2d_2)S'_2 + \frac{1}{B_2^3}(P_1 + P_2)S_2,$$

$$E\Delta_3 = \frac{1}{B_3^2}(P_1d_1 + P_2d_2 + P_3d_3)S'_3 + \frac{1}{B_3^3}(P_1 + P_2 + P_3)S_3,$$

etc. etc.

The S'_n's and S_n's for all the joints should first be separately computed by aid of Table VIII, and then combined in the last written expressions, above.

Similarly,

$$E\theta_1 = \frac{1}{B_1}(P_1 d_1 S'''_1) + \frac{1}{B_1^2} P_1 S''_1,$$

$$E\theta_2 = \frac{1}{B_2}(P_1 d_1 + P_2 d_2) S'''_2 + \frac{1}{B_2^2}(P_1 + P_2) S''_2,$$

$$E\theta_3 = \frac{1}{B_3}(P_1 d_1 + P_2 d_2 + P_3 d_3) S'''_3 + \frac{1}{B_3^2}(P_1 + P_2 + P_3) S''_2,$$

etc. etc.

The S'''_n's and S''_n's, should be separately computed as for S'_n and S_n, in advance.

The resulting expressions for $E\Delta_1$, $E\Delta_2$, etc., with $E\theta_1, E\theta_2$, etc., will each now consist of n terms; for example, $E\Delta_3$ or $E\theta_3$ will each have 3 terms, involving P_1, P_2, and P_3 with numerical coefficients.

Example:

The computation for $E\Delta_2$, for instance, may be arranged as follows:

There will be nine terms involving (I), (II), (III), (IV), (V), and (VI), of Table VIII. Assume $l_2 = 3.86$, $B_2 = 0.09285$, $R_2 = 350$.

Substituting in Eq. (13),

$$E\Delta_n = \frac{1}{B^2}(P_1 d_1 + P_2 d_2 + \ldots + P_n d_n)\Big[-\frac{2100 - 4l_n}{350}(\mathrm{I})$$
$$-(700 - l_n)(\mathrm{II}) + (\mathrm{IV}) + \frac{443.80 - 0.268 l_n}{350}(\mathrm{V})$$
$$+\frac{1656.2 - 3.732 l_n}{350}(\mathrm{VI})\Big] + \frac{1}{B^3}(P_1 + P_2 + \ldots + P_n)$$
$$\times\Big[(4200 - 6l_n)(\mathrm{I}) - (\mathrm{III}) - (2100 - 1.268 l_n)(\mathrm{V})$$
$$-(2100 - 4.722 l_n)(\mathrm{VI})\Big] \quad \ldots\ldots\ldots \quad (15)$$

S'_n and S_n comprise the quantities within the two sets of brackets and their determination is illustrated in Table IX. For $E\Delta_2$, S'_2 and S_2 are here shown, for example, the quantities of Joint 2, in Table VIII, being used for (I), (II), etc.

TABLE IX

COMPUTATION OF S'_2 AND S_2

For $E\Delta_2$

$-\frac{2100-15.44}{350}\times(\mathrm{I})_2$	=	-0.0092501	For S'_2
$-(200-3.86)\times(\mathrm{II})_2$	=	-1.2964911	
$+(\mathrm{IV})_2$	=	$+1.0303393$	
$+\frac{443.80-0.268\times3.86}{350}(\mathrm{V})_2$	=	-0.0000061	
$+\frac{16562-3.732\times3.86}{350}\times(\mathrm{VI})_2$	=	-0.0000850	
S'_2	=	-0.2754930	
$+(4200-6\times3.86)\times(\mathrm{I})_2$	=	$+6.4870502$	For S_2
$-(\mathrm{III})_2$	=	-5.032262	
$-(2100-1.268\times3.86)(\mathrm{V})_2$	=	$+0.0101005$	
$-(2100-4.732\times3.86)\ (\mathrm{VI})_2$	=	$+0.0377100$	
S_2	=	$+1.5025987$	

$$\frac{1}{B_2^2}(P_1d_1+P_2d_2)S'_2+\frac{1}{B_2^3}(P_1+P_2)S_2,$$

assuming, for sake of explanation, values for d_1, d_2, of 25.5384 and 41.5384, respectively, or

$$115.994478(P_1 \times 25.5384 + P_2 \times 41.5384)(-0.2754930)$$
$$+1249.21924(P_1 + P_2)(+1.5025987)$$
$$E\Delta_2 = -816.0966P_1 - 1327.3873P_2 + 1877.0752(P_1 + P_2)$$
$$= 1060.9786P_1 + 549.6879P_2.$$

$E\theta_2$ should be similarly calculated.

A separate sheet should be devoted to the computation of each $E\Delta_n$ and each $E\theta_n$, so that ready reference may be made thereto.

The Δ's and θ's, having been thus computed, should next be collected according to formula (12), or (12*a*) or (12*b*), as applicable, and by aid of Table VII.

To this end, the following form of tabulation (Table X) for each D_n will prove convenient, for obtaining and summing coefficients of the unknowns, P, assuming, for example, five load-points.

Application of Eq. (8). (Arch Equation.)—Having written the expressions for cantilever deflections (ED_5, ED_4, . . . ED_1) as indicated in Table X, it remains to construct the equations for the deflections of the corresponding arches, by the application of Eq. (8).

The tabulation of Table XI will expedite the formation of the arch deflection equations, shown in the last column of Table XI.

Eq. (8) is based upon the assumption of ends fixed for the arch. Limitations to which this assumption may be subject have been pointed out previously in relation to arch action as a whole; but the usual excavations for the dam into the rock sides of the gorge or canyon, justify to some extent, this assumption.

For the determination of the angles ϕ_n of the various

TABLE X

COMPUTATION OF "*ED*'s."

	P_1	P_2	P_3	P_4	P_5	Line No.
$ED_5=\Delta_5$ ..... =	+ 80.40272	+ 70.49011	+51.90397	+ 29.60060	+ 6.95648	1
$[\Delta_4$ .... =	+477.07724	+404.02633	+267.05587	+102.69132]		2
$\Delta_4+\Delta_5$ ..... =	+557.47996	+474.51644	+318.95984	+132.29192	+ 6.95648	3
$(d_5-d_4)\theta_5$ ..... =	+349.10199	+304.97893	+222.42670	+123.36403	+22.78425	
ED_4 ..... =	+906.58195	+779.49537	+541.38655	+255.65595	+29.74073	4
etc.	etc.	etc.	etc.	etc.	etc.	
$[\Delta_1$ ..... =	+ 407.05927]					14
$\Delta_1+\Delta_2+\Delta_3+\Delta_4+\Delta_5$ .... = (from above)	+2827.53097	+1635.89320	+ 573.89073	+132.29192	+ 6.95648	
$(d_2-d_1)\theta_2$ ..... =	+1254.03198	+ 512.89922				15
$(d_3-d_1)\theta_3$ ..... =	+2295.13600	+1686.83527	+ 546.26408			16
$(d_4-d_1)\theta_4$ ..... =	+2337.95013	+1954.58152	+1235.76066	+373.17574		17
$(d_5-d_1)\theta_5$ ..... =	+1132.00332	+ 989.19980	+ 721.44148	+400.13150	+73.90077	18
ED_1 ..... =	+9846.65241	+6779.40901	+3077.35694	+905.59916	+80.85725	19

arch laminæ, an average depth of excavation (often about 15 feet) may be assumed on the profile of the site. This may be transferred, by means of excavation contours, to the plan of site and points thereon determined at the levels of the assumed load-points, or joints. These points are taken on the foundations, so that they are approximately midway between the upstream and downstream faces of the dam at their respective elevations in rock. A line on the plan through these points is the line of the profile for the axes of the various arches, whence the central angles ϕ_n are readily obtainable. They are to be entered in the second column of Table XI.

The trigonometric function of ϕ_n, CC_c, may be read directly from Fig. 28, p. 149.

TABLE XI

ARCH EQUATIONS

Joint No.	ϕ_n	$\frac{l}{2}$	R_n	$R_n - \frac{l_n}{2} = r_n$	$\frac{R_n}{r_n}$	$Q_n - P_n$ (tons)	g_n
1	59°	1.185	350	348.815	1.003397	$0.56 - P_1$	6.0
2	etc.	etc.	etc.	etc.	etc.	etc.	etc.
etc.							

Joint No.	$q_n = \frac{R_n}{r_n}\left(\frac{Q_n - P_n}{g_n}\right)$ Eq. (7)	Function of $\phi_n = CC_c$	$\frac{r_n^2}{l_n}$	ED_c (Eq. (8))
1	$0.09365 - 0.16723P_1$	1.0952	51338.35621	$5265.5657 - 9402.7982P_1$
2	etc.	etc.	etc.	etc.
etc.				

Simultaneous Equations Formed.—Equating each arch deflection to its corresponding cantilever deflection (for instance, ED_1 of Table X with ED_c of Table XI, for Joint No. 1) and reducing, eliminates the quantity E_1 between the two, and the simultaneous equations (involving P with numerical coefficients) as many as there are joints, resulting, may be tabulated as indicated in Table XII. These simultaneous equations are to be solved for the values of the quantities P, in tons.

TABLE XII

FINAL SIMULTANEOUS EQUATIONS

Coefficients of Quantities P.

Joint No.	P_1	P_2	P_3
1	+19249.45065	+67709.40901	+3077.35694
2	etc.	etc.	etc.
3	etc.	etc.	etc.
4	etc.	etc.	etc.
5	+80.40272	+70.49011	+ 51.90397

Joint No.	P_4	P_5	=
1	+905.59916	+80.85725	+5265.56566
2	etc.	etc.	etc.
3	etc.	etc.	etc.
4	etc.	etc.	etc.
5	+29.60060	+215.65413	29155.10921

For nearly all of the above computations, a calculating machine will be found most expeditious.

The final results may be tabulated in some such form as in Table XIII, in which loads are in tons and intensities of stress, in tons per square foot.

TABLE XIII

RESULTS

Joint No.	Elev.	Total Load Concentration (Q_n).	Load Assumed by		Percent. Arch Action.	Maximum Stresses in		
			Cantilever (P_n).	Arch ($Q_n - P_n$).		Horizontal Plane (Cantilever)		Vertical Radial Planes (Arch).
						Up-stream.	Down-stream.	
1	96.0	0.56	−1.87	+ 2.43	435			
2	80.0	12.55	−4.11	+16.66	133	+13.9	−11.7	etc.
3	50.0	etc.	+4.14		etc.	etc.	etc.	etc.
etc.								
Base	14.0	etc.						

Cantilever Stresses.—To get the resulting maximum stress intensities in any horizontal joint of the cantilever, combine the average intensity of stress on that joint due to the weight of the superimposed masonry with the stress intensity due to the moment of that masonry plus the moments of the cantilever loading, P, all moments taken about the center of gravity of the horizontal section of the cantilever at the given joint. For calculating the position of the centroid of the horizontal section of the cantilever at any joint of length l_n, referred to the down-stream edge of the given joint, use the expression $\frac{l_n}{3}\left(\frac{3R_n - l_n}{2R_n - l_n}\right)$. Or, this resultant moment of the forces about the centroid of the given horizontal section of the cantilever may be thus calculated and entered into the well-known expression, $M = \frac{kI}{d_1}$, as written on page 29, Eq. (18), or using here C for d_1 and S_d (down-stream intensity) and S_u (up-stream intensity) for k, to prevent confusion, the last expression may be written in the form

$$S = \frac{MC}{I}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14a)$$

By substituting l_n for Bx in Eq. (1), for I, reducing, and inserting in Eq. (14a), the expression for S_d results:

$$S_d = \frac{12M_nR_n(3R_n - l_n)}{l_n^2[6R_n(R_n - l_n) + l_n^2]}, \quad . \quad . \quad . \quad (14b)$$

and

$$S_u = -\frac{(3R_n - 2l_n)}{(3R_n - l_n)}S_d. \quad . \quad . \quad . \quad . \quad . \quad (14c)$$

It should be remembered that to the stress intensities found by Eqs. (14b) and (14c) should be added the average intensity of stress due to the superimposed masonry, to get the resultant intensity, as previously stated. (In computing the weight of masonry, it will be sufficiently close to use an average horizontal base in each case, of rectangular section in lieu of the trapezoidal bases.)

Arch Stresses.—To get the resulting maximum stress intensity on the vertical radial planes, at any joint-level, n, at the crown, the expression for M_c should be employed, viz.:

$$M_c = \frac{q_n l_n^2}{12}\left(\frac{\phi_n - \sin\phi_n}{\phi_n}\right)\left(\frac{2\sin\phi_n}{3\phi_n + \sin\phi_n\cos\phi_n - 4\sin\phi_n}\right) \quad (14d)$$

and the maximum stress in the arch, S_a, is:

$$S_a = \frac{6M_c}{l_n^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (14e)$$

The stress resulting from Eq. (14e), for any level, n, must be combined with the axial arch thrust (S_t) as found by Eq. (14f):

$$S_t = \frac{q_n}{l_n}\left(R_n - \frac{l_n}{2}\right). \quad . \quad . \quad . \quad . \quad . \quad (14f)$$

If the stress is desired at any other point than at the crown, the expression M_ϕ, in Eq. (12), of Appendix II, should be employed.

Arch Dam of Rectangular Cross-section. Special Case.—The foregoing, it will be remembered, is applicable to a dam of any cross-section, with load points assumed at will.

In case it is desirable to investigate a dam of rectangular vertical cross-section, the load points being chosen at equal intervals a vertical distance, a, apart, with constant moment of inertia for the horizontal section of cantilever, the following expressions, Nos. 15 to 18, apply.

These are derived in a similar way to those that have preceded. Some slight approximation may have to be made in computing the total load concentrations, if they be considered a constant distance apart, but usually a dam of rectangular cross-section is not very high and this approximation will not be serious.

Loads, etc., are located with reference to the top of the dam in this case, with $d_n = na$.

$$M_n = P_1(x-a) + P_2(x-2a) + \ldots + P_n(x-na). \quad (15)$$

$$EI\Delta_n = \frac{a^3}{6}\Big\{[2+3(n-1)]P_1 + [2+3(n-2)]P_2 + \ldots + [2+3(n-n)]P_n\Big\}. \quad (16)$$

$$EI\theta_n = \frac{a^2}{2}\Big\{[1+2(n-1)]P_1 + [1+2(n-2)]P_2 + \ldots + [1+2(n-n)]P_n\Big\}. \quad (17)$$

$$D_n = (\Delta_n + \Delta_{n+1} + \Delta_{n+2} + \ldots + \Delta_m) + a[\tan\theta_{n+1} + 2\tan\theta_{n+2} + 3\tan\theta_{n+3} + \ldots + (m-n)\tan\theta_m]. \quad (18)$$

As before θ may be written for $\tan\theta$.

Expressions (16), (17) and (18) for the rectangular section of arched dam correspond to expressions (13), (14) and (12), respectively, and fulfill the same offices. Eq. (8) still applies, for the arches to be considered.

The derivation of Eqs. (16) and (17) are comparatively simple and are outlined in Appendix I.

For further references to methods of analysis of arch dams, the reader is referred to Transactions of the American Society of Civil Engineers, Vol. 53.

CHAPTER VIII

RECENT CONSIDERATIONS OF THE CONDITION OF STRESS IN MASONRY DAMS

CONSIDERABLE discussion has been raised within the past few years, by criticisms being leveled at the present general procedure in the design of high masonry dams. This has properly perhaps, been more pronounced abroad than in this country, since the matter may be said to have been precipitated by the publication of a paper by Mr. L. W. Atcherley of London University, "On Some Disregarded Points in the Stability of Masonry Dams."*

It is the purpose to outline the analysis as presented there, and to call attention to some of the discussion which followed, in order to indicate the status of the theory involved in the design of such structures.

The paper referred to takes exception to current practice in regard to the matter of design and indicates a need for both revision and extension in the analysis, and then, supplementing the generally accepted ideas as to the distribution of normal stress on horizontal planes, by an assumption as to the shear on these planes, proceeds to show that peculiar and unexpected conditions arise.

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* Dept. of Applied Mathematics, University College, University of London. Drapers' Company Research Memoirs. Technical Series II.

It is a fact that, owing 1st, to the manner in which masonry structures are built, i.e., of a mixture of stone and cement, and 2d, to the nature of the sections at the springings or areas of support, it is practically impossible to apply to them the general theory of elastic bodies. Consequently, the treatment as it is employed to-day has been developed, but only by the use of certain assumptions which it may be shown are not precisely exact.

The basis of the present investigation rests upon the four following formulæ, in which the usual distribution of normal unit stress on horizontal planes is accepted, but to which is added an assumed condition as to the distribution of horizontal shear.

$$cd = \frac{1\zeta^2}{12}, \quad \ldots \ldots \ldots \quad (1)$$

$$C_{\max} = \frac{Q}{A}\left(1 + \frac{6c}{\zeta}\right), \quad \ldots \ldots \ldots \quad (2)$$

$$T_{\max} = \frac{Q}{A}\left(\frac{6c}{\zeta} - 1\right), \quad \ldots \ldots \ldots \quad (3)$$

$$S = \frac{3}{2}\frac{P}{A}\left(1 - \frac{4y^2}{\zeta}\right). \quad \ldots \ldots \ldots \quad (4)$$

c = distance along the horizontal joint from the centroid to the point of application of the resultant.

d = distance along the horizontal joint from the centroid to the point locating the neutral axis.

ζ = length of the horizontal joint.

$C_{\max}$ = maximum compressive stress on the joint.

$T_{\max}$ = maximum tensile stress on the joint.

Q = vertical component of the resultant force acting on the joint.

A = area of the joint.

S = shear at any point y in the joint.

P = total shear on the joint.

y = the distance from the centroid to any point on the joint.

With regard to Eq. (4) it may be stated that it has not heretofore been customary to consider the distribution of shearing stress on horizontal joints. But, if the distribution of normal stresses may be assumed to be represented by Eqs. (1), (2), and (3), with equal validity for the usual types of dam, may the shear at any point be assumed to be represented by Eq. (4). It is believed by Mr. Atcherley that these equations more nearly express the conditions of equilibrium in a dam than the usual ones do, even though the latter tacitly assume the first three by imposing the condition of the middle third, and use a friction condition, instead of one for shear as expressed by Eq. (4).

In reference to this friction factor there may be some question of doubt, since M. Levy* prescribes an angle of 30° for masonry on masonry, while Rankine gives 36°; on the other hand, examination of dams actually built frequently shows the angle to lie somewhere between the above values.

But whatever its exact value, the friction condition leaves some doubt as to the actual distribution of shear

* "La Statique graphique." IVe Partie, 'Ouvrages en Maçonnerie," page 92.

over a horizontal joint, the variation of which must be known, in order to determine the tensile and compressive stresses *on the vertical sections of the tail* (*i.e., downstream portion*) *of the dam.* In consequence of this the parabolic law as expressed by Eq. (4) has been assumed and will later be shown to be more nearly correct than any other hypothesis.

According to the author there is no reason whatever why dams should be tested solely by taking horizontal cross-sections, and asserting that the line of resistance must lie in the middle third, while the stresses across the vertical sections of the tail are absolutely neglected. If the former condition is valid, then no dam ought to be passed unless it can be shown also that there is no tension of any serious value across vertical cross-sections of the tail, parallel to the length of the structure. It is believed that a great number of dams as now designed will be found to have very substantial tension in these sections and this, in the opinion of the author, is a source of weakness in dam construction which has not been properly considered and allowed for.

If the problem is to be solved on the assumption that a dam is an "isotropic and homogeneous" structure, the general equations for the stresses can be determined only by the following considerations:

(*a*) The normal and shearing stresses on the horizontal top and curved flank, i.e., downstream face, are both zero.

(*b*) The normal stress on the battered front or upstream face is equal to the water pressure, and the shear is zero, and

(*c*) Either the stresses or the shifts must be supposed given over the base.

It follows at once from this that Eqs. (1), (2), and (3) are not *absolutely* true, but that the shear is fairly closely represented by Eq. (4).

As far as the present investigation is concerned, however, the enquiry is not as to the validity of the usual treatment; it is obviously faulty. But it is the purpose to try to indicate that, supposing it to be correct, its present *partial* application, i.e., to horizontal joints only, involves the serious, and, it is believed, often dangerous, neglect of large tension across the vertical sections.

To justify the above statement, two model dams of wood were employed for experimental purposes, the cross-sections being identical, and agreeing with that of a dam actually constructed. One of these models was subdivided into horizontal strata to study the effect on such planes, and the other into vertical longitudinal strata, for a similar purpose. The application of the loading was such that it approximated as closely as possible the conditions obtaining in an actual dam. The general conclusions from these experiments were that:

(*a*) The current idea that the critical sections of a dam are the horizontal ones is entirely erroneous. A dam collapses first by the tension on the vertical sections of the tail.

(*b*) The shearing of the vertical sections over each other follows immediately on this opening up by tension.

(*c*) It is probable that the shear on the horizontal sections is also a far more important matter than is usually supposed.

It follows consequently, that keeping the line of resistance within the middle third of the horizontal sections is by no means the hardest part of dam design. It would be surprising if, with all the labor spent on this point, the bulk of existing dam constructions are not, for masonry, under very considerable tension, i.e., a tension across the vertical sections which has been hitherto disregarded.

It is proposed therefore to lay it down as a rule for the construction of future dams that the stability of the dam from the standpoint of the vertical sections must be considered in the *first* place. If this be satisfactory, it is believed that the horizontal sections will be found to be stable, but of course the latter must be independently investigated.

The above conclusions were apparently verified by a combined analytical and graphical treatment in which the algebraical analysis will here be considered first.

Denoting the total vertical force acting on a horizontal joint by Q_0, and the total horizontal force acting over the same by P_0, under the assumption that the reservoir is full, the variation of the normal pressure on the joint may be represented by the straight line of Eq. (1).

If the resultant pressure on the joint be assumed to cut it at the extremity of the middle third, then according to the previous notation, d will have a value of $\frac{1}{3}\frac{b^2}{c}$, provided $2b$ is the length of the joint. This indicates that the line representing the variation of normal pressure over the joint intersects it at the upstream edge, and any vertical between it and the joint itself will represent the normal pressure at that point where the vertical is erected.

Denoting this by y, it may be termed "the vertical height-giving pressure," and may also be expressed in terms of height of masonry, if the factors upon which it depends are expressed in cubic feet of masonry.

Again, we may write an equation of the downstream face, with respect to the same joint so long as that face is a straight line, by making $y' = mx$.

Evidently then if this latter line, and the one indicating the variation of pressure over the base, be referred to the same origin, the tip of the tail, the difference in areas included between each and the base will represent the total upward force, in cubic feet of masonry, acting over any assumed portion, "x" of the joint, measured from the tail.

Representing this upward force by F_1 its point of application may be easily determined, while the shear may be written as F_2, being regulated by Eq. (4).

As F_1 and F_2 thus give all the external forces, considering a wedge-shaped piece of dam bounded by the downstream face, a vertical and a horizontal plane, the total shear on the vertical plane must equal F_1 and the total thrust F_2, since these internal stresses are held in equilibrium by external forces. Thus F_1 equals the total shear on the vertical section, at a distance x from the tip of the tail, while F_2 equals the total horizontal thrust over the same.

If y be expressed in terms of x, and locate the point on the successive vertical planes through which the resultant acts, then the equation will represent the line of resistance on these vertical planes. It is found to be an hyperbola.

Considering the stresses on the vertical sections, it is

found: First, that the maximum shear may be properly represented by $\frac{3}{2}$ the mean value, and may be so arranged as to be expressed in terms of F_1 and mx. Such an equation, representing a straight line, immediately shows the necessity of thickening the tip of the tail which, as a matter of fact, is the usual procedure in actual design. Second, the line representing the maximum tensile stress may be shown to vary as a parabola whose axis is vertical.

When the downstream face ceases to be linear, it becomes necessary to apply a graphical solution for the determination of the stresses. This it is unnecessary to reproduce here, but the curves may be said to indicate the following results:

(1) That the line of resistance for the vertical sections lies outside the middle third for rather more than half the vertical sections. In other words, these sections are subjected to tension.

(2) That the tensile stresses in the tail are, for masonry, very serious, amounting to nearly 10 tons per square foot at the extreme tip, and to 6 tons per square foot after we have passed the vertical section, where the strengthening of the tail has ceased.

(3) That the maximum shearing stresses amount to 6 tons per square foot at the tip of the tail and 5 tons per square foot after we have passed the vertical section, where the strengthening of the tail has ceased. No undue importance should be laid on the actual values of these "maximum" shears on the vertical sections however, as they are obtained from the mean shears by using the round multiplier 1.5. This round number is assumed because the maximum is certainly greater than the mean shear. The

actual distribution of shear on the vertical sections has not been discussed. It could, of course, be found from that on the horizontal sections, if the latter were really known with sufficient accuracy, by the equality of the shears on two planes at right angles. It is sufficient to show that the *mean* shears on the vertical sections appear to be higher than those on the horizontal section, and thus indicate that the parabolic distribution applied to sections some way above the base, probably *under-estimates* the maximum shearing in the dam.

In other words: *Whether the test is made by the line of resistance lying outside the middle third, or by the existence of serious tensile stresses, or by the magnitude of the mean shearing stresses, the vertical sections are critical for the stability in a far higher degree than the horizontal sections.*

In a well-designed dam, all the conditions for stability of the horizontal sections may have been satisfied, yet if the very same conditions be applied to the vertical sections not one of them will be found to be satisfied. It seems accordingly very unsatisfactory that the current tests for stability should, if they are legitimate, be applied to the horizontal instead of to the far more critical vertical sections. In the case of the latter they fail completely; and if higher tension and shear are to be allowed in the vertical sections, then it is absurd to exclude them in the case of the horizontal sections. It is maintained by the author that the current treatment of dams is fallacious, for it screens entirely the real source of weakness, namely, in the first place the tension, and in the second place the substantial shear, in the vertical sections, and this at dis-

tances from the tail far beyond the usual tail-strengthening range.

Nor do these theoretical results stand unverified by experiment; they are absolutely in accord with the experiments on the model dams. These collapsed precisely as might have been expected from the above investigation, i.e., the dam with vertical sections gave long before the dam with horizontal sections. The former collapsed by opening up of the joints by tension towards the tail, followed almost immediately by a shear of the whole structure. In the case of the horizontally stratified dam, the collapse, which occurred much later, was by shear of the base, followed almost simultaneously by a shear of one or more of the horizontal sections.

The question then arises as to how far the previously assumed distribution of shear affects the main features of the results, and so the other extreme was taken, i.e., uniform shear, and the effect determined.

This distribution must be further from the actual than the first hypothesis, yet it is still found:

(1) That the line of resistance falls well outside the middle third for about half the dam.

(2) That there exist considerable tensions, 3 to 4 tons per square foot, in the masonry.

(3) That the average shearing stresses on the vertical sections are greater than on the horizontal sections. As a result of this extreme case, it is believed that the real distribution of shear over the base, whatever it may be, must lead us to a line of resistance lying well outside the middle third, and to tensions amounting to something between 5 and 10 tons per square foot.

From these investigations the author concludes as follows:

(1) The current theory of the stability of dams is both theoretically and experimentally erroneous, because:

(*a*) Theory shows that the vertical and not the horizontal sections are the critical sections.

(*b*) Experiment shows that a dam first gives by tension of the vertical sections near the tail.

(2) An accepted form of cross-section is shown to be stable as far as the horizontal sections are concerned, but unstable by applying the same conditions of stability to the vertical sections.

(3) The distribution of shear over the base must be more nearly parabolic than uniform, but as no reversal of the statements follows in passing from the former to the latter extreme hypothesis, it is not unreasonable to assume the former distribution will describe fairly closely the facts until we have greater knowledge.

(4) In future it is held that in the first place masonry dams must be investigated for the stability of their vertical sections. If this be done it is believed that most existing dams will be found to fail, if the criteria of stability usually adopted for their horizontal sections be accepted. This failure can be met in two ways:

(*a*) By a modification of the customary cross-section. It is probable that a cross-section like that of the Vyrnwy dam would give better results than more usual forms.

(*b*) By a frank acceptance that masonry, if carefully built, may be trusted to stand a definite amount of tensile stress. It is perfectly idle to assert that it is absolutely necessary that the line of resistance shall lie in the middle

third for a horizontal treatment, when it lies well outside the middle third for at least half the dam for a vertical treatment.

Immediately upon the publication of the preceding results, Sir Benjamin Baker undertook some experiments of a like nature.* The models employed by him were

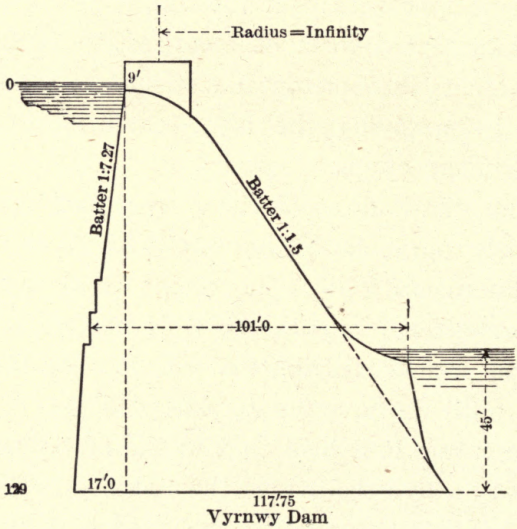

FIG. 29.

made of ordinary jelly however, and included not only the transverse section of the dam itself, but the rock upon which it rested as well. It is shown in the figure.

The horizontal and vertical lines drawn on the sides of the model were for the purpose of detecting any distortion that might result through the application of

* Vol. 162, page 120. Minutes of Proceedings of the Institution of Civil Engineers.

pressure. These pressures were applied against both the upstream face and the floor of the reservoir, as it was believed that, while according to the theory of the middle third there could be no tension in the heel, nevertheless for the case of reservoir full, fairly severe tension in the masonry might thus be caused.

The experiments indicated that the distribution of shearing stress in the plane of the base, i.e., where the

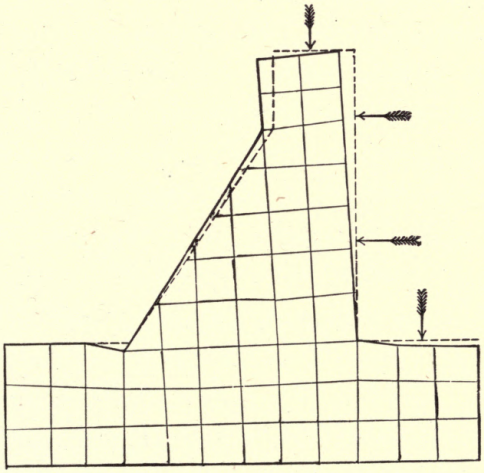

FIG. 30.

dam met the rock, was more nearly uniform than parabolic, and that the strain extended into the rock for a distance equal to about half the height of the dam before it became undetectable. To solve the complete problem, therefore, it would be necessary to consider the elasticity of the rock on which the dam rested. Partially as a result of these and the previous experiments, it may be pointed out in passing, the proposed increase in elevation of the Assouan dam, whereby the capacity of the reservoir

would have been considerably augmented, was indefinitely postponed.

The new feature in Atcherley's analysis is that, even though the condition of "no tension in a horizontal joint" is satisfied, dangerous tensions may be shown to exist across vertical planes.* In connection with this consider, for example, a section of the dam *ABC*, which is triangular in profile, and construct *BEC* so that the ordinates represent the variation of the unit normal stress over the horizontal joint *BC*.

Taking a vertical section *IK* in which *J* locates the centroid, the forces to the left are the upward pressure acting over *BK*, tending to cause rotation in a clock-wise manner and thus produce tension at *K*, and two counteracting forces tending to neutralize this pressure: the weight of the portion *BKI* and the horizontal shearing force acting along *BK*. The resultant effect of all three will be tension at *K*, provided the rotation is right-handed, with a consequent splitting along the vertical plane *IK*.

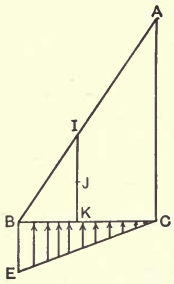

FIG. 31.

In view of the fact that the horizontal shear is present as a factor, it is necessary to determine its distribution, and this Prof. W. C. Unwin undertook to do.† Instead however, of accepting the distribution in accordance with Atcherley's assumptions, an analysis was attempted by

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* "Engineering," Vol. 79, page 414.

† "Engineering," Vol. 79, page 513. "Note on the Theory of Unsymmetrical Masonry Dams," by W. C. Unwin.

which the shear might be actually calculated, and in doing so attention was called to the fact that the accepted theory of dam design is incomplete in just that feature, since it fails to consider the rate of change in the horizontal shear.

In any analysis the fundamental assumption must be made that a masonry dam is a homogeneous-elastic solid, and, while it is not absolutely essential that no tension exist at any point in the cross-section, yet it seems desirable that there should be none at the upstream face of horizontal joints.

It may be said therefore, that for a more exact analysis the problem resolves itself into one of the determination of shear on horizontal planes, and Prof. Unwin suggests as follows, a method of procedure by which this may be accomplished:

If, as in the figure, we assume a dam of triangular section, in which AB is some horizontal joint, other than the base, and C its centroid, then Q will represent the water thrust, P the weight of masonry, and R their resultant.

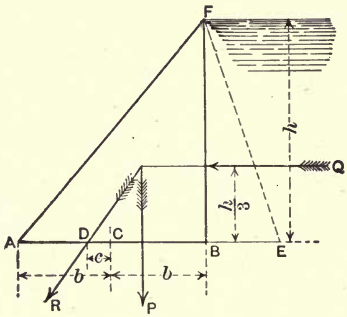

FIG. 32.

In agreement with the ordinary theory we may write the well-known formula for the unit normal pressure on a horizontal joint, at any point x, measured from A, as follows:

$$P_n = \frac{1}{2b}\left(1 + \frac{3c(b-x)}{b^2}\right). \quad . \quad . \quad . \quad . \quad (5)$$

For the horizontal shear we must proceed further. Consider, therefore, the forces to the left of HK in Fig. 33, we have (1) the vertical pressure on AK, (2) the weight of AHK, and (3) the shear acting along AK. It is evident that the difference between (1) and (2) represents the total vertical shear on HK.

If, therefore, the figure $ALMB$ represent, in masonry units, the distribution of normal stress on AB, as given by Eq. (1), then $ALTH$ will, in like manner, represent the above-mentioned total vertical shear on HK.

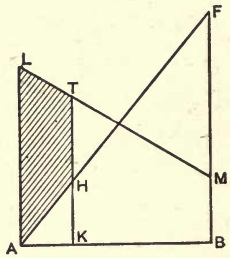

FIG. 33.

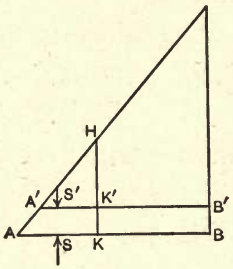

FIG. 34.

Consider now a second section $A'B'$, a small distance z above AB; the total shear on HK' may then be found as before. Denoting the former by S, and the latter by S', then $S-S'$ equals the total shear on KK', which, when divided by z, will give the intensity of vertical shear at K, and consequently the intensity of horizontal shear at the same point.

Since all the forces to the left of HK are now known, the normal stress on that plane may be found, and from it we may readily determine whether tension or compression exists at K.

At the base these results would be much modified,

because of the discontinuity of form, which, in the opinion of Prof. Unwin, places the exact determination of the stresses beyond the power of mathematics. The author believes the effect of the rock into which the dam is built is to reduce the variation of stress which would otherwise exist.

In a subsequent paper,* giving a complete demonstration of the preceding analysis as applied to a masonry dam of triangular cross-section, it is found that the distribution of shear on a plane horizontal joint may be represented by a right triangle whose base is the length of the joint and whose vertex is perpendicularly below the downstream edge. The figure illustrates the variation of normal stress and shear on AB; the lines of resistance for both vertical and horizontal planes; and the centers of gravity of the sections above the successive horizontal joints.

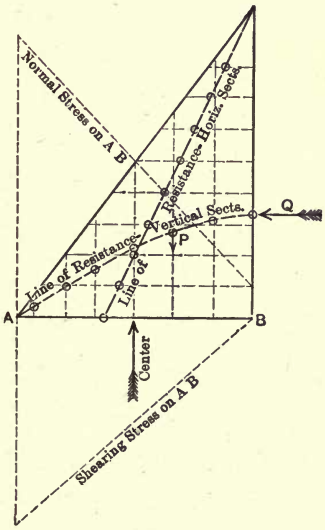

FIG. 35.

Consequently the total normal or shearing stress on any part of AB is equal to the area between that part and the line of normal stress or the line of shearing stress.

If the upward reactions and the weights of the dam to

* "Engineering," Vol. 79, page 593. "Further Note on the Theory of Unsymmetrical Masonry Dams." W. C. Unwin.

the left of each vertical section be combined with the shears T, acting along AB, the resultants will cut the vertical sections at points shown on the line of resistance for these vertical sections. As this line lies wholly within the middle third, there can be no tension on any vertical section.

The total compressive stress on any vertical section at its lower edge will therefore be:

$$\frac{T}{y}\left(1+\frac{6z}{y}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

where T is the shear on the horizontal plane from the toe to the vertical section taken, y the height of the vertical section, and z the distance from the center of the vertical section to the point of application of the resultant forces on that section.

Near the upstream toe the plane on which the greater principal stress acts is found to be vertical while near the downstream toe it approaches the horizontal. The stresses are all compressive and on the water face the compressive stress is at all points equal to the water pressure at that point.

The above analysis is simply an application to vertical sections of the method now accepted as applicable to the horizontal planes and is a possible solution, since the distribution of shear is known. It differs from Atcherly's method in the fact that the latter assumes the usual distribution of *normal* stress, together with *a parabolic variation for the horizontal shear.* This latter hypothesis the author thinks inconsistent with the previous one.

Further investigations by Prof. Unwin* on dams of various sections lead to the following conclusions:

(1) For a rectangular dam the distribution of shearing stress on horizontal planes may be represented by the ordinates of a parabola.

(2) For a triangular dam, the distribution may be represented by the ordinates of a triangle with the apex below the downstream toe.

(3) For a dam with vertical upstream face and curved downstream face the distribution may be represented by a figure consisting of a parabola superposed on a triangle.

(4) For a dam with rectangular base the distribution is represented by a parabola.

Following the results of the experimental investigations of Atcherley and Baker, several other papers of a like nature appeared in the Minutes of Proceedings of the Institute of Civil Engineers, Vol. 162. The first of these to be considered here is that by Sir John Walter Ottley and Arthur William Brightmore, entitled, "Experimental Investigations of the Stresses in Masonry Dams subjected to Water-Pressure."

In presenting this paper, the authors drew attention to the fact that until the publication of Mr. Atcherley's results, the question of dam design had been accepted as settled, and that his memoir had had the effect of reopening the entire subject of the distribution of stress in structures of this class.

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* "Engineering," Vol. 79, page 825. "On the Distribution of Shearing Stress in Masonry Dams." Prof. W. C. Unwin.

It was also pointed out that tension was found by him to exist on vertical planes near the outer toe, whether the distribution of shearing stress over the base was assumed to be uniform or to vary according to the parabolic law.

Considering a transverse section of a dam, the authors argued that, whatever the distribution of shear over the base might be, it must follow some other law near the top, since the conditions in these higher levels are radically different from those existing in the lower, where the dam is fixed to the foundation, and where the water pressure ceases abruptly.

The investigation was therefore undertaken, at least in part, to determine the distribution of shear on horizontal planes in the higher levels of the dam and to see how it varied from that at the base; and it might be stated here that it was found to be uniform in the latter plane but to vary uniformly from zero at the heel to a maximum at the toe in the higher levels, the change from the one condition to the other being gradual. It will be shown that it is near the inner toe rather than near the outer toe that tension may be anticipated.

The model dams were triangular in section, made from a kind of modeling clay called "plasticine," and so proportioned that the resultant pressure on the base cut that plane at the downstream extremity of the middle third.

For purposes of observation the sections were placed between vertical sides of plate glass, upon which vertical and horizontal lines had been etched, corresponding to similar lines on the model, so that any displacement in

the latter might be noted by comparison with the former. Pressure was applied, by means of a thin rubber bag containing water which was made to fit the frame. Though the water was allowed to act over a period of 33 days, after the elapse of one week a crack was noticed at the upstream toe, running downward and at an angle of about 45°. At the end of the longer period an examination showed that in the neighborhood of the base the displacement of the vertical lines was such as to make them all about equally inclined, thus indicating a uniform intensity of shear on that section, while in the higher levels and near the outer portion of the dam the lines became more inclined as the elevation increased, indicating that the intensity of shear increased also as the top was approached.

Turning to the horizontal lines in the model for the purpose of discovering the method of distribution of normal stress, it was found that they were curves at the base, sloping downward from the inner toe to a point about two-thirds the distance to the outer toe, then remaining fairly level until almost reaching the downstream face, when they finally bent up slightly. In the higher levels, however, these lines gradually developed a uniform slope running from the inner to the outer toe.

An investigation of the shearing stresses on vertical planes requires that, to draw the line representing the intensity of normal reaction at the base the following facts must be considered:

(1) The total normal reaction equals the weight of the dam.

(2) Since the resultant pressure on the base acts at

one-third the width from the outer toe, the moment of the reaction stresses about this point must be zero.

(3) The intensity of the reaction at the outer toe must equal the intensity of the shearing stress in the vertical plane multiplied by the ratio of the height to the base of the dam.

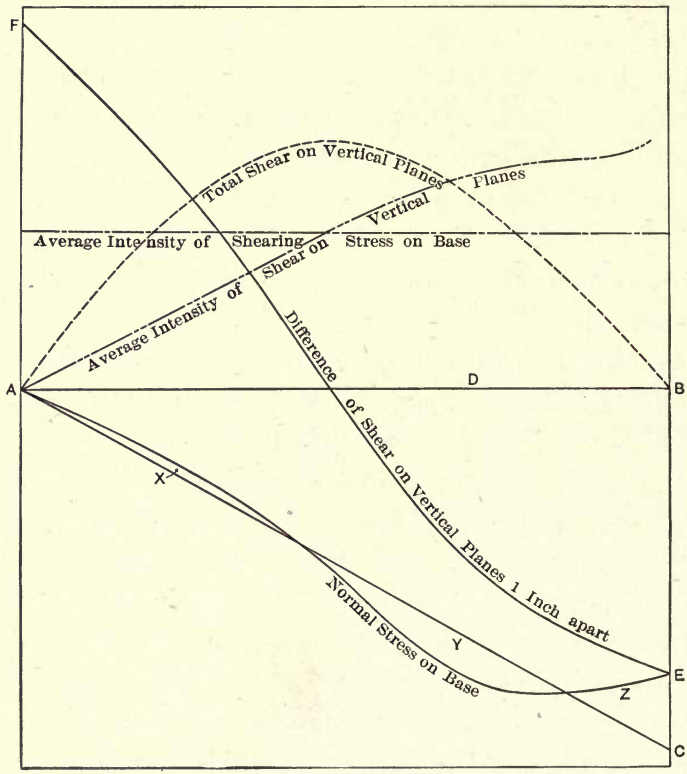

FIG. 36.

Referring to the figure: *AB* represents the base of the dam, and *BC* twice the average intensity of normal stress on *AB*. *AC* is then drawn; consequently *ABC* represents the total normal stress on *AB*, or the weight of the structure.

If AE, on the other hand, represents the actual intensity of normal reaction over AB, then for (1) to hold true the area Y must equal the areas $(x+z)$ and if (2) is to hold, the moments of x, y, and z, about D (equal to $\frac{1}{3}AB$ from B), must be zero; also for (3) to be satisfied, BE must equal the limiting value of shearing stress in a vertical plane near the toe, multiplied by the height and divided by the base of the dam.

From these considerations AE may be fitted in by trial till it is found to satisfy all of the above conditions.

Dividing the cross-sections into vertical strips 1 inch wide we may properly consider the equilibrium of each such strip. Evidently the difference between the weight of each strip and the normal reaction on the base is equal to the difference in shear on the two adjacent vertical planes, and if in the figure these weights be plotted upward from AE, the curve FE will result. Furthermore, both the curves for "total shear on vertical planes" and "average intensity of shear on vertical planes" may now be drawn, whereupon it is evident to what extent the average intensity of shear on vertical planes varies, and how it compares with the average intensity on the base.

Since the shear on horizontal and vertical planes at any one point is equal, and the shear on the base is practically constant, it follows that above the base the shear on horizontal or vertical planes is small near the heel while in the outer half above the base it increases as the outer edge is approached; in fact it increases from zero at the heel to a maximum at the toe. These facts show that the shearing stresses to be provided for are those existing

in the higher levels and near the toe, and not those at the base.

In considering the effect of shear on the base, neglecting the "fixing" at that level, we may assume that the reaction stress and that due to the weight of a strip, is constant over each inch. They then act at the middle of each strip; and, taking these points successively as centers, the difference of the moments of the horizontal pressures on the vertical sides of the strip, it is evident, will equal the sum of the shearing stresses on the same vertical sides multiplied by ½ inch.

This makes possible the determination of the moment of the horizontal pressures on each vertical strip.

The horizontal shear on each inch of base being the difference between the horizontal pressures acting on the two vertical sides, the latter may be determined as soon as their points of application are given. As these points are known for the innermost and outermost strip, an easy curve may be drawn which will approximately locate the other points and thus give the desired heights. From these results it may be shown that the shearing stress on the base increases from practically zero at the inner toe to a point near the center of the base and then remains fairly constant.

The modification of this distribution, due to the fixing of the dam to its base, must, on the other hand, be considered. The water tends to cause a maximum pressure and displacement at the inner face, which diminishes to zero at the outer. As the dam is fixed, this displacement is prevented, thus inducing corresponding shears, and the effect of this conflicting condition, with that previously

shown to exist, causes a nearly uniform shear over the base.

Further evidence of uniform shear on the base was obtained as follows: The models, after being subjected to water pressure, showed cracks which appeared at the inner toe, the angles which these made with the horizontal steadily diminishing as the base was decreased in width from a maximum of 45° for the widest base used to 25° for the narrowest.

The variation of these inclinations corresponded closely with the computed directions, on the assumption that the shear was uniform over the base and the experiments therefore strongly support the inference that shear over the base is uniformly distributed.

It was shown by means of the models that there are tensile stresses on other than horizontal planes passing through the inner toe. The models indicated this by cracking, even when the back was sloped away from the vertical so as to cause vertical pressure and hence compression on the upstream face.

The impossibility of tension on vertical planes near the outer toe may be shown by means of the following equation for principal stress:

$$f = \frac{p + p' \pm \sqrt{(p + p')^2 - 4(pp' - q^2)}}{2} \quad . \quad . \quad . \quad (7)$$

where compressions are plus and tensions are minus. When $pp' > q^2$ at any point, there can be no tension at that point, since under the above conditions both principal stresses will be compression and hence stresses on all other planes passing through that point will be compression

also. This condition may be shown to exist near the outer toe, and hence no tension can act across any vertical plane in that position.

For example consider the equilibrium of a wedge of unit length cut off by a vertical plane near the toe.

p' = intensity of pressure normal to a vertical plane at base.

p = intensity of reaction normal to the base.

q = intensity of shearing stress.

Then $p'h = qb$ or,

$$p' = q\frac{b}{h} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

The weight of the particle is negligible because it varies with h^2.

Since the resultant stress must be parallel to the outer face it follows that,

$$p = q\frac{h}{b} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Multiplying (8) by (9) there results

$pp' = q^2$ at outer toe. But p has been shown to increase for some distance from outer toe and the point of application of p' becomes relatively lower as the inner toe is approached and since the average pressure is constant it follows that p increases as the distance from the outer toe increases and hence in the vicinity of the outer toe pp' is greater than q^2 and consequently there can be no tension in that neighborhood. This was checked by the behavior of the models.

It is a fact that in dam work the normal stress is the only one specified, whereas the absolute maximum is about 50 per cent greater.

The conclusions reached from this set of experiments follow:

(1) If a masonry dam be designed on the assumption that the stresses on the base are uniformly varying and that the stresses are parallel to the resultant force acting on the base, the actual normal and shearing stresses on both horizontal and vertical planes would be less than those provided for.

(2) There can be no tension on any planes near the outer toe.

(3) There will be tension on certain planes other than the horizontal near the inner toe, and the maximum intensity of such tension in the foundation being generally equal to the average intensity of shearing stress on the base, and the inclination of its plane of action being about 45°; and its maximum intensity in the dam above the base about ½ the above amount and acting on a plane less inclined to the horizontal.

The investigation undertaken by Mr. Hill * for "The Determination of the Stresses on any Small Element of Mass in a Masonry Dam," are on the other hand purely analytical in character, being directed toward a solution of (1) the vertical, (2) horizontal, and (3) tangential shearing forces acting on the faces and along the edges of such an element.

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* Minutes of Proceedings of the Inst. of C. E., 72.

In this analysis, there is first expressed a perfectly general formula for C (the distance of the load point from the center of the joint), and two other general formulæ for the pressures p_1 and p_2 in terms of the total load and C from its above value, where p_1 is the minimum and p_2 the maximum pressure. For the pressure p at any point x on the joint of length b the following equation is used:

$$p = p_1 + \frac{x}{b}(p_2 - p_1) \quad . \quad . \quad . \quad . \quad . \quad (10)$$

Up to this point the analysis is identical with the general procedure of investigation, which assumes that the horizontal pressures are proportional to the vertical, and does not analyze the shear.

Citing Prof. Unwin, the author states tnat the former "suggested that the shearing stress at any point might be found by considering the difference between the total net vertical reactions (between that point and either face) along two horizontal planes a unit's distance apart, and has applied the principle by the use of algebraical methods." Mr. Hill, on the contrary, employs the calculus to obtain more rigorous results.

The procedure follows: Consider any point distant x from the inner toe and on the lower of two horizontal planes, a unit's distance apart. The total vertical reaction is then $\int_0^x p dx$. Subtracting the weight of masonry resting on this portion of the horizontal joint, and denoting the difference by r we have an expression for the "net vertical reaction." If this value of r be differentiated

with respect to h, the distance between the two horizontal planes, the change in the reaction will be obtained, and this change or difference is the vertical shearing stress at the point located by x. It is also, therefore, the horizontal shear at the same point, which we may denote by q.

If q be integrated with respect to x, between the limits of x and b, the resulting expression will give the entire horizontal shear between such limits on the joints in question. Represent this by Qx.

To find the horizontal pressure intensity, we have but to consider the above integration. This shear must be resisted by the material along the vertical section at x. Similarly the total shear on a plane a differential distance below the last must be resisted by the vertical section at x, differing in height from the former by dh. Consequently the differential of Qx with respect to $h = p'$ will represent the horizontal pressure intensity at point x. These expressions for p, p' and q therefore give respectively the values of the vertical pressure intensity, horizontal pressure intensity, and shearing force acting on a unit element of mass.

Cain * presents a treatment of this matter, which, while presenting no new features, is strictly arithmetical in character, and in that respect at least differs from the preceding. Its purpose, as Hill's, is to determine the amount and distribution of stress at any point in a masonry

* Wm. Cain, M. Am. Soc. C. E., Trans. Am. Soc. C. E., Vol. 64, page 208.

dam, on the assumption that the law of the trapezoid represents the variation of pressure on horizontal joints.

The analysis finally establishes formulæ for (1) the normal unit stress at any point in a horizontal joint, (2) the normal unit stress on a vertical plane at any point of a horizontal joint, (3) the unit shear on either horizontal or vertical planes at any point of a horizontal joint, and at the same time indicates the method of determining the maximum and minimum normal stresses and the planes on which they act.

The solutions are only approximate, but the results are found to be close enough for the purpose.

Before proceeding it may be advisable to review certain features involved in a consideration of the stresses in a masonry dam which Prof. Cain presents in a very satisfactory manner.

1. It will be evident from an examination of the figure that the intensities of shear on two planes at right angles to each other are equal. For, in the elementary cube under consideration, the weight may be neglected, since it is an infinitesimal of the third order, while the opposing normal forces balance as the cube is reduced in size.

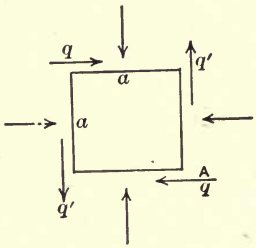


FIG. 37.

For equilibrium then, $q \cdot a \cdot a = q' \cdot a \cdot a$, or $q = q'$ and, because each side is a differential quantity, it may be assumed that the values q and q' represent the average unit shear on the respective faces. As a consequence they are equal to the shear at any point, for example A, of the particle.

2. In a triangular element of the dam, Fig. 38, at the down-stream edge, and of unit's length, the forces acting are those shown. Because it is an element we may neglect the weight, and therefore, if p' is the normal intensity of stress on a vertical plane, p the normal intensity of stress on a horizontal plane, and q the shear intensity, for

$$\Sigma V = 0, pb = qa \quad \text{or} \quad p = q\frac{a}{b} \quad \text{and} \quad q = p \tan \phi. \quad . \quad (11)$$

for

$$\Sigma H = 0, \; p'a = qb \quad \text{or} \quad p' = q\frac{b}{a}. \quad . \quad . \quad (12)$$

then

$$p' = p \tan^2 \phi,$$

or,

$$pp' = q^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

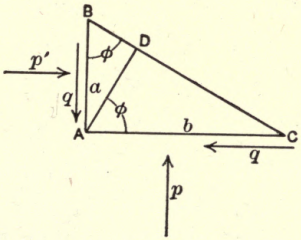


FIG. 38.

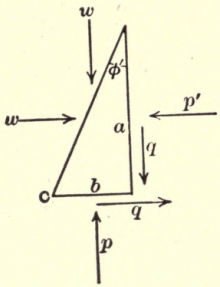


FIG. 39.

3. The same analysis may be applied to an element at the inner face, Fig. 39, where ϕ' is the inclination to the vertical; but, for the reservoir full, the intensity of water pressure, horizontal or vertical, at c, and in this case represented by w, must be taken into account.

Under these circumstances,

$$pb = qa + wb, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

and

$$p'a = qb + wa, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

$$\therefore \quad p = q \cot \phi' + w, \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

and

$$p' = q \tan \phi' + w. \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

When, as is usually the case, the vertical component of water pressure acting along the back is neglected, the above equations become,

$$p = q \cot \phi', \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

$$p' = q \tan \phi' + w, \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

$$\therefore \quad q = p \tan \phi', \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

and

$$p' = \tan^2 \phi' + w. \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

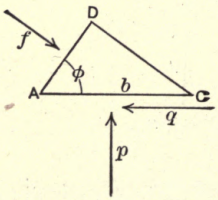


FIG. 40.

4. If an element at the down stream face be again considered, since the shear on the outer face DC is zero, that on a plane AD perpendicular to DC, must be zero also, and hence the stress d on AD is wholly normal.

The total pressure on AD is therefore,

$$f \cdot AD = f \cdot b \cos \phi. \quad . \quad . \quad . \quad . \quad (22)$$

The vertical component of this is $f \cdot b \cos^2 \phi$, because $\Sigma V = 0$,

$$pb = f \cdot b \cos^2 \phi, \quad . \quad . \quad . \quad . \quad . \quad (23)$$

or,

$$f = \frac{p}{\cos^2 \phi} = p \sec^2 \phi, \quad . \quad . \quad . \quad . \quad (24)$$

which is the maximum intensity of normal stress at the outer face.

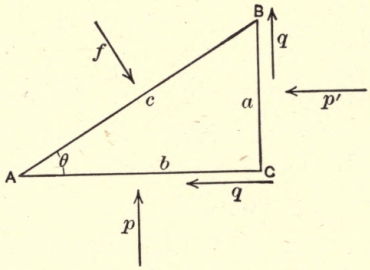


FIG. 41.

5. To determine the planes of principal stress, i.e., planes upon which the stress is wholly normal, and also the intensity of that stress, we may assume the conditions indicated in the figure.

The total stress on c then is fc; its vertical component $fc \cos \theta = fb$, and its horizontal component $fc \sin \theta = fa$.

When $\Sigma V = 0$ and $\Sigma H = 0$,

$$fb = pb + qa; \qquad \therefore \; f - p = q \tan \theta, \quad . \quad . \quad (25)$$

$$fa = qb + p'a; \qquad \therefore \; f - p = q \cot \theta, \quad . \quad (26)$$

The difference of these two equations gives,

$$p-p'=q\,(\cot\theta-\tan\theta)=q\,\frac{1-\tan^2\theta}{\tan\theta} \quad . \quad (27)$$

$$\therefore\ \tan 2\,\theta=\frac{2\tan\theta}{1-\tan^2\theta}=\frac{2q}{p-p'}. \quad . \quad . \quad (28)$$

This equation gives a plane upon which there is none but normal stress.

To determine f, multiply equation (25) by (26).

$$(f-p)\,(f-p')=q^2, \quad . \quad . \quad . \quad . \quad (29)$$

whence,

$$f=\tfrac{1}{2}[p+p'\pm\sqrt{(p+p')^2-4\,(pp'-q^2)}]. \quad . \quad . \quad (30)$$

This will give two values of f, which correspond to the two principal planes of stress, the stress being compressive when f is positive, and tensile when f is negative. There can be no tension when $pp'\geqq q^2$.

Determination of the vertical unit stress at any point of a horizontal plane joint: From the law of the trapezoid, we have the pressures at the upstream and downstream toes represented respectively as follows:

$$p_2=\frac{4b-6EC}{b^2}W, \quad . \quad . \quad . \quad . \quad . \quad (31)$$

$$p_1=\frac{4b-6CB}{b^2}W. \quad . \quad . \quad . \quad . \quad . \quad (32)$$

The resultant is supposed to act within the middle third. If x' represent any point along EB, measured from E, then p, the pressure at x, is given by,

$$p=p_2+\frac{p_1-p_2}{b}x', \quad . \quad . \quad . \quad . \quad (33)$$

while the total normal stress from E to x' is, by integration,

$$P = p_2 x' + \frac{p_1 - p_2}{2\,b} x'^2. \quad . \quad . \quad . \quad . \quad (34)$$

To find the unit shear on vertical or horizontal planes, we have but to consider a slice of dam between two horizontal joints one foot apart, extending from the inner to the outer face, a distance x along the lower joint. (The back is supposed to slope .02 feet for each foot in height).

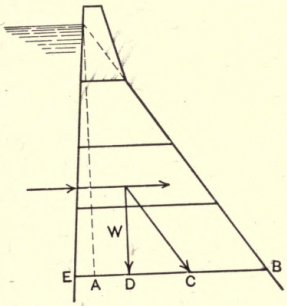


FIG. 42.

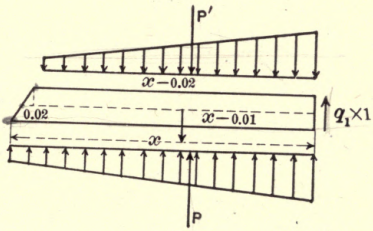


FIG. 43.

The vertical forces acting are:

(1) A uniformly varying stress on the upper joint acting downward.

(2) The same on the lower joint acting upward.

(3) The weight of the strip.

(4) The shear on the vertical face at x.

For equilibrium,

$$q_1 = P' - P + (x - 0.01). \quad . \quad . \quad . \quad . \quad (35)$$

P' and P may be obtained as indicated in the previous demonstration.

The above value of q_1 is the average unit shear at the depth taken, but a similar value q_2 may be determined at a depth one foot below. Under these circumstances $\frac{q_1+q_2}{2}$ is the average of the two, and may be said to be approximately equal to the shear at the depth of the joint between the two slices.

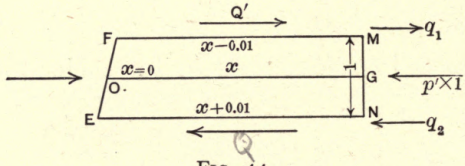


FIG. 44.

To find the normal unit stress on a vertical plane, a similar section to that just used may be employed; but the horizontal components are now to be equated for equilibrium.

Let h = the horizontal water pressure at the assumed depth.

Q' = total shear on upper face.

Q = total shear on lower face.

p' = average normal stress.

q_1 and q_2 = the intensities of horizontal shear at the points indicated.

Q' and Q may be found by integrating q_1 and q_2 between the proper limits.

$$\therefore \quad p' = h + Q' - Q. \quad . \quad . \quad . \quad . \quad . \quad (36)$$

This value of p' is assumed as the average intensity on the vertical plane and as the unit intensity on the

vertical plane at a point midway between the two horizontal planes.

Three general formulæ may be written for p, q, and p' which, it has been suggested, be put in the following form:

$$p = a + bx, \quad \ldots \ldots \ldots \quad (37)$$

$$q = c + dx + ex^2, \quad \ldots \ldots \quad (38)$$

$$p' = f + gx + hx^2 + jx^3. \quad \ldots \ldots \quad (39)$$

APPENDIX I

DERIVATION OF CANTILEVER EQUATIONS

THE expression for the moment of inertia, I, of the horizontal cross-section of the cantilever contained between two vertical radial planes of the arched dam, and the two fundamental equations for deflection represented by Δ_n and θ_n, will now be derived. The derivation of the expressions for these last two quantities for the special case of an arched dam of rectangular vertical cross-section will also be indicated.

Derivation of the expression for the moment of inertia, I, of the horizontal cross-section of the cantilever at the distance x, below the given origin, or Eq. (1) of page 146:

$$I = \frac{6R_n^2B^3x^3 - 6R_nB^4x^4 + B^5x^5}{36R_n(2R_n - Bx)}.$$

Referring to the nomenclature of page 145, and to Fig. 27, it is evident that $l = Bx$. (Cf. Fig. 45 and Fig. 46.)

If we assume two vertical radial planes to intersect the dam, 1 foot apart at the up-stream face, the plan of the section to be considered will be indicated by the shaded area in Fig. 45. It will be sufficiently exact, however, to substitute the chords for the curves themselves as limiting this area. Furthermore, where the up-stream face of the section under consideration is battered, the average value,

between the two successive load points, may be assumed for the up-stream radius, R_n, as local variation in this respect

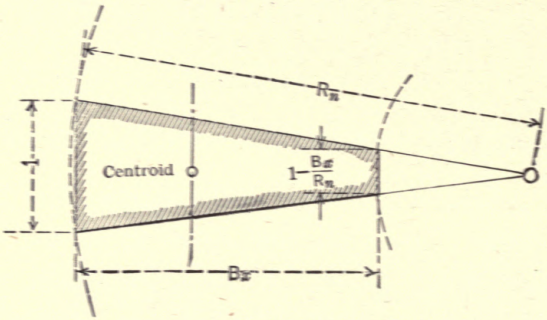


FIG. 45.

has been treated as negligible in the derivation of the general expression for I.

The resulting expression will then be transformed by the substitution of like terms into one corresponding to

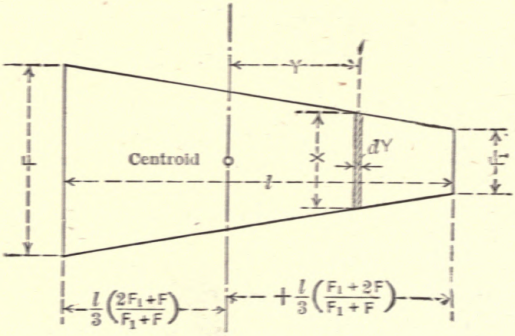


FIG. 46.

the nomenclature of Fig. 45, which will be the desired Eq. (1).

The reciprocal of Eq. (1) will then be modified by factoring its denominator, in order to render it integrable in further operations where it may occur.

The general expression for the moment of inertia, I, of Fig. 46, is

$$I=\int XY^2dY. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (a)$$

in which

$$X=\frac{2(F^2+FF_1+F_1^2)}{3(F_1+F)}-\frac{F-F_1}{l}Y \quad . \quad . \quad . \quad (b)$$

Substituting this value of X in Eq. (a) and designating the limits of integration as

$$Y=+\frac{l}{3}\left(\frac{F_1+2F}{F_1+F}\right)$$

and

$$Y=-\frac{l}{3}\left(\frac{2F_1+F}{F_1+F}\right),$$

there results:

$$I=\frac{2(F^2+F_1F+F_1^2)}{3(F_1+F)}\int_{-\frac{l}{3}\left(\frac{2F_1+F}{F_1+F}\right)}^{+\frac{l}{3}\left(\frac{F_1+2F}{F_1+F}\right)} Y^2dY$$

$$-\frac{F-F_1}{l}\int_{-\frac{l}{3}\left(\frac{2F_1+F}{F_1+F}\right)}^{+\frac{l}{3}\left(\frac{F_1+2F}{F_1+F}\right)} Y^3dY. \quad (c)$$

Performing the above indicated integrations,

$$I=\frac{2(F^2+F_1F+F_1^2)}{3(F_1+F)}\left[\frac{Y^3}{3}\right]_{-\frac{l}{3}\left(\frac{2F_1+F}{F_1+F}\right)}^{+\frac{l}{3}\left(\frac{F_1+2F}{F_1+F}\right)}-\frac{F-F_1}{l}\left[\frac{Y^4}{4}\right]_{-\frac{l}{3}\left(\frac{2F_1+F}{F_1+F}\right)}^{+\frac{l}{3}\left(\frac{F_1+2F}{F_1+F}\right)}.$$

Completing the above, gives

$$I=\frac{l^3}{972(F_1+F)^4}\Big\{8(F^2+F_1F+F_1^2)[(F_1+2F)^3+(2F_1+F)^3]$$
$$-3(F-F_1)[(F_1+2F)^4-(2F_1+F)^4]\Big\},$$

which may be reduced to

$$I = \frac{l^3(F_1{}^2 + 4F_1F + F^2)}{36(F_1 + F)}. \quad . \quad . \quad . \quad . \quad . \quad (d)$$

To transform Expression (d) into the form of Eq. (1), substitute in (d), 1 for F, $1 - \frac{Bx}{R_n}$ for F_1, and Bx for l, as indicated in Fig. 45. There will then result

$$I = \frac{6R^2{}_nB^3x^3 - 6R_nB^4x^4 + B^5x^5}{36R_n(2R_n - Bx)}, \quad . \quad . \quad . \quad . \quad (1)$$

which is Eq. (1) of page 207.

To render $\frac{1}{I}$ *integrable by factoring.*

From the above equation we have

$$\frac{1}{I} = \frac{36R_n(2R_n - Bx)}{B^5x^3\left[x^2 - \frac{6R_n}{B}x + 6\left(\frac{R_n}{B}\right)^2\right]}, \quad . \quad . \quad . \quad . \quad (e)$$

By factoring $x^2 - \frac{6R_n}{B}x + 6\left(\frac{R_n}{B}\right)^2$ in the denominator of (e), into the typical factors, $(x+a)(x+b)$, there results, after reduction, the following integrable form:

$$\frac{1}{I} = \frac{36R_n(2R_n - Bx)}{B^5x^3\left[x - (3 + \sqrt{3})\frac{R_n}{B}\right]\left[x - (3 - \sqrt{3})\frac{R_n}{B}\right]}. \quad . \quad . \quad (f)$$

Development of the equations for Deflection in the Cantilever for $E\Delta_n$ and $E\theta_n$:

Eqs. (9) and (10) of page 150 are the fundamental differential expressions from which are derived Eqs. (13) and (14), for the cantilever.

As there stated, the expression for I and that for M are employed by substituting them in Eqs. (9) and (10) and integrating. These operations are indicated below.

Derivation of $E\Delta_n$.

Eq. (9) of page 150 is

$$d\Delta = \frac{M(x-d)dx}{EI} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Eq. (11) of page 150 is

$$M_n = (P_1 + P_2 + P_3 + \ . \ . \ . \ + P_n)x$$
$$-(P_1d_1 + P_2d_2 + P_3d_3 + \ . \ . \ . \ + P_nd_n). \quad (11)$$

Eq. (*f*) of p. 210, Appendix, may be written

$$\frac{1}{I} = \frac{36R_n\left(\frac{2R_n}{B} - x\right)}{B^4x^3\left[x - (3+\sqrt{3})\frac{R_n}{B}\right]\left[x - (3-\sqrt{3})\frac{R_n}{B}\right]} \quad . \quad . \quad . \quad (f')$$

The general integral of Eq. (9) may be written

$$E\int d\Delta = \int \frac{M}{I}(x-d)dx. \quad . \quad . \quad . \quad . \quad (g)$$

Substituting the expressions for M_n and $\frac{1}{I}$ in Eq. (*g*) and reducing, gives

$$E\Delta_n = \frac{36R_n}{B^4}\int_{d_n}^{d_{n+1}} \left\{ \frac{\frac{2R_n}{B}(P_1d_1+P_2d_2+\ldots+P_nd_n)d_n}{x^3\left[x-(3+\sqrt{3})\frac{R_n}{B}\right]\left[x-(3-\sqrt{3})\frac{R_n}{B}\right]}dx \right.$$

$$-\frac{\frac{2R_n}{B}[(P_1+P_2+\ldots+P_n)d_n+(P_1d_1+P_2d_2+\ldots+P_nd_n)]+(P_1d_1+P_2d_2+\ldots+P_nd_n)d_n}{x^2\left[x-(3+\sqrt{3})\frac{R_n}{B}\right]\left[x-(3-\sqrt{3})\frac{R_n}{B}\right]}dx$$

$$+\frac{\frac{2R_n}{B}(P_1+P_2+\ldots+P_n)+(P_1d_1+P_2d_2+\ldots+P_nd_n)+(P_1+P_2+\ldots+P_n)d_n}{x\left[x-(3+\sqrt{3})\frac{R_n}{B}\right]\left[x-(3-\sqrt{3})\frac{R_n}{B}\right]}dx$$

$$\left. -\frac{P_1+P_2+\ldots+P_n}{\left[x-(3+\sqrt{3})\frac{R_n}{B}\right]\left[x-(3-\sqrt{3})\frac{R_n}{B}\right]}dx \right\}. \qquad (h)$$

To integrate Eq. (h) with respect to x, between the limits d_{n+1} and d_n, it will be necessary, first, to determine the integrals for the four fractions containing dx in the numerator of each, and the denominators as written above, functions of descending powers of x, beginning with x^3, as a factor for the first denominator.

These separate integrations may be accomplished by expanding each fraction into a series of partial fractions by the method of undetermined coefficients and then integrating each term of the series. This will result in Eqs. (i), (j), (k), and (l). These equations will serve for the derivation of $E\theta_n$, as well as for $E\Delta_n$.

According to the theorem of undetermined coefficients, there may be written for the first fraction of Eq. (h) expanding in ascending powers of x and distinguishing

B, the coefficient, by a vertical letter from B, the batter, an inclined letter,

$$\frac{dx}{x^3\left[x-(3+\sqrt{3})\frac{R_n}{B}\right]\left[x-(3-\sqrt{3})\frac{R_n}{B}\right]}=\frac{\mathbf{A}dx}{x}+\frac{\mathbf{B}dx}{x^2}+\frac{\mathbf{C}dx}{x^3}$$

$$+\frac{\mathbf{D}dx}{x-(3+\sqrt{3})\frac{R_n}{B}}+\frac{\mathbf{E}dx}{x-(3-\sqrt{3})\frac{R_n}{B}}.$$

Clearing of fractions, collecting the terms in the second member involving like powers of x, equating coefficients of like powers of x in the two members and solving, gives

$$\mathbf{A}=+\frac{5}{36}\left(\frac{B}{R_n}\right)^4,$$

$$\mathbf{B}=+\frac{1}{6}\left(\frac{B}{R_n}\right)^3,$$

$$\mathbf{C}=+\frac{1}{6}\left(\frac{B}{R_n}\right)^2,$$

$$\mathbf{D}=-\frac{5-3\sqrt{3}}{72}\left(\frac{B}{R_n}\right)^4,$$

$$\mathbf{E}=-\frac{5+3\sqrt{3}}{72}\left(\frac{B}{R_n}\right)^4.$$

The integral of this first fraction may thence be written

$$\int_{d_n}^{d_{n+1}}\frac{dx}{x^3\left[x-(3+\sqrt{3})\frac{R_n}{B}\right]\left[x-(3-\sqrt{3})\frac{R_n}{B}\right]}$$

$$=+\frac{5}{36}\left(\frac{B}{R_n}\right)^4\int_{d_n}^{n+1}\frac{dx}{x}+\frac{1}{6}\left(\frac{B}{R_n}\right)^3\int_{d_n}^{d_{n+1}}\frac{dx}{x^2}$$

$$+\frac{1}{6}\left(\frac{B}{R_n}\right)^2\int_{d_n}^{d_{n+1}}\frac{dx}{x^3}-\frac{5-3\sqrt{3}}{72}\left(\frac{B}{R_n}\right)^4\int_{d_n}^{d_{n+1}}\frac{dx}{x-(3+\sqrt{3})\frac{R_n}{B}}$$

$$-\frac{5+3\sqrt{3}}{72}\left(\frac{B}{R_n}\right)^4\int_{d_n}^{d_{n+1}}\frac{dx}{x-(3-\sqrt{3})\frac{R_n}{B}}.$$

The definite integral of which is

$$\int_{d_n}^{d_{n+1}} \frac{dx}{x^3\left[x-(3+\sqrt{3})\frac{R_n}{B}\right]\left[x-(3-\sqrt{3})\frac{R_n}{B}\right]}$$

$$= +\frac{5B^4}{36R_n^4}\log_\epsilon\frac{d_{n+1}}{d_n}+\frac{B^3(d_{n+1}-d_n)}{6R_n^3d_nd_{n+1}}+\frac{B^2(d^2_{n+1}-d_n^2)}{12R_n^2d_n^2d^2_{n+1}}$$

$$-\frac{(5-3\sqrt{3})B^4}{72R_n^4}\log_\epsilon\frac{d_{n+1}-(3+\sqrt{3})\frac{R_n}{B}}{d_n-(3+\sqrt{3})\frac{R_n}{B}}$$

$$-\frac{(5+3\sqrt{3})B^4}{72R_n^4}\log_\epsilon\frac{d_{n+1}-(3-\sqrt{3})\frac{R_n}{B}}{d_n-(3-\sqrt{3})\frac{R_n}{B}}. \quad . \quad . \quad . \quad . \quad (i)$$

The second fraction of Eq. (h), by the method of undetermined coefficients, may be expanded, thus:

$$\frac{dx}{x^2\left[x-(3+\sqrt{3})\frac{R_n}{B}\right]\left[x-(3-\sqrt{3})\frac{R_n}{B}\right]}=\frac{\mathbf{A}dx}{x}+\frac{\mathbf{B}dx}{x^2}$$

$$+\frac{\mathbf{C}dx}{x-(3+\sqrt{3})\frac{R_n}{B}}+\frac{\mathbf{D}dx}{x-(3-\sqrt{3})\frac{R_n}{B}},$$

whence there follow the values:

$$\mathbf{A}=+\frac{1}{6}\left(\frac{B}{R_n}\right)^3,$$

$$\mathbf{B}=+\frac{1}{6}\left(\frac{B}{R_n}\right)^2,$$

$$\mathbf{C}=+\frac{2\sqrt{3}-3}{36}\left(\frac{B}{R_n}\right)^3,$$

$$\mathbf{D}=-\frac{2\sqrt{3}+3}{36}\left(\frac{B}{R_n}\right)^3.$$

The integral of the second fraction may then be written:

$$\int_{d_n}^{d_{n+1}} \frac{dx}{x^2\left[x-(3+\sqrt{3})\frac{R_n}{B}\right]\left[x-(3-\sqrt{3})\frac{R_n}{B}\right]}$$

$$=\frac{1}{6}\left(\frac{B}{R_n}\right)^3\int_{d_n}^{d_{n+1}}\frac{dx}{x}+\frac{1}{6}\left(\frac{B}{R_n}\right)^2\int_{d_n}^{d_{n+1}}\frac{dx}{x^2}$$

$$+\frac{2\sqrt{3}-3}{36}\left(\frac{B}{R_n}\right)^3\int_{d_n}^{d_{n+1}}\frac{dx}{x-(3+\sqrt{3})\frac{R_n}{B}}$$

$$-\frac{2\sqrt{3}+3}{36}\left(\frac{B}{R_n}\right)^3\int_{d_n}^{d_{n+1}}\frac{dx}{x-(3-\sqrt{3})\frac{R_n}{B}}.$$

The definite integral of which is

$$\int_{d_n}^{d_{n+1}} \frac{dx}{x^2\left[x-(3+\sqrt{3})\frac{R_n}{B}\right]\left[x-(3-\sqrt{3})\frac{R_n}{B}\right]}$$

$$=+\frac{B^3}{6R_n^3}\log_e\frac{d_{n+1}}{d_n}+\frac{B^2(d_{n+1}-d_n)}{6R_n^2d_nd_{n+1}}$$

$$+\frac{(2\sqrt{3}-3)B^3}{36R_n^3}\log_e\frac{d_{n+1}-(3+\sqrt{3})\frac{R_n}{B}}{d_n-(3+\sqrt{3})\frac{R_n}{B}}$$

$$-\frac{(2\sqrt{3}+3)B^3}{36R_n^3}\log_e\frac{d_{n+1}-(3-\sqrt{3})\frac{R_n}{B}}{d_n-(3-\sqrt{3})\frac{R_n}{B}}\quad\ldots\ldots\quad(j)$$

Similarly, for the third fraction of Eq. (h),

$$\frac{dx}{x\left[x-(3+\sqrt{3})\frac{R_n}{B}\right]\left[x-(3-\sqrt{3})\frac{R_n}{B}\right]}=\frac{\mathbf{A}dx}{x}$$

$$+\frac{\mathbf{B}dx}{x-(3+\sqrt{3})\frac{R_n}{B}}+\frac{\mathbf{C}dx}{x-(3-\sqrt{3})\frac{R_n}{B}}.$$

Proceeding as before, there result,

$$\mathbf{A}=+\frac{1}{6}\left(\frac{B}{R_n}\right)^2,$$

$$\mathbf{B}=+\frac{\sqrt{3}-1}{12}\left(\frac{B}{R_n}\right)^2,$$

$$\mathbf{C}=-\frac{\sqrt{3}+1}{12}\left(\frac{B}{R_n}\right)^2.$$

The integral of this third fraction follows:

$$\int_{d_n}^{d_{n+1}}\frac{dx}{x\left[x-(3+\sqrt{3})\frac{R_n}{B}\right]\left[x-(3-\sqrt{3})\right]\frac{R_n}{B}}$$

$$=\frac{1}{6}\left(\frac{B}{R_n}\right)^2\int_{d_n}^{d_{n+1}}\frac{dx}{x}+\frac{\sqrt{3}-1}{12}\left(\frac{B}{R_n}\right)^2\int_{d_n}^{d_{n+1}}\frac{dx}{x-(3+\sqrt{3})\frac{R_n}{B}}$$

$$-\frac{\sqrt{3}+1}{12}\left(\frac{B}{R_n}\right)^2\int_{d_n}^{d_{n+1}}\frac{dx}{x-(3-\sqrt{3})\frac{R_n}{B}}.$$

The definite integral of which is:

$$\int_{d_n}^{d_{n+1}} \frac{dx}{x\left[x-(3+\sqrt{3})\frac{R_n}{B}\right]\left[x-(3-\sqrt{3})\frac{R_n}{B}\right]}$$

$$= +\frac{B^2}{6R_n^2}\log_\epsilon \frac{d_{n+1}}{d_n} + \frac{(\sqrt{3}-1)B^2}{12R_n^2}\log_\epsilon \frac{d_{n+1}-(3+\sqrt{3})\frac{R_n}{B}}{d_n-(3+\sqrt{3})\frac{R_n}{B}}$$

$$-\frac{(\sqrt{3}+1)B^2}{12R_n^2}\log_\epsilon \frac{d_{n+1}-(3-\sqrt{3})\frac{R_n}{B}}{d_n-(3-\sqrt{3})\frac{R_n}{B}} \quad \ldots\ldots \quad (k)$$

And for the fourth fraction of Eq. (h),

$$\frac{dx}{\left[x-(3+\sqrt{3})\frac{R_n}{B}\right]\left[x-(3-\sqrt{3})\frac{R_n}{B}\right]}$$

$$= \frac{\mathbf{A}dx}{x-(3+\sqrt{3})\frac{R_n}{B}} + \frac{\mathbf{B}dx}{x-(3-\sqrt{3})\frac{R_n}{B}}.$$

From which:

$$\mathbf{A} = +\frac{\sqrt{3}}{6}\left(\frac{B}{R_n}\right),$$

$$\mathbf{B} = -\frac{\sqrt{3}}{6}\left(\frac{B}{R_n}\right).$$

The integral of this fourth fraction is:

$$\int_{d_n}^{d_{n+1}} \frac{dx}{\left[x-(3+\sqrt{3})\frac{R_n}{B}\right]\left[x-(3-\sqrt{3})\frac{R_n}{B}\right]}$$

$$= +\frac{\sqrt{3}}{6}\left(\frac{B}{R_n}\right)\int_{d_n}^{d_{n+1}} \frac{dx}{x-(3+\sqrt{3})\frac{R_n}{B}}$$

$$-\frac{\sqrt{3}}{6}\left(\frac{B}{R_n}\right)\int_{d_n}^{d_{n+1}} \frac{dx}{x-(3-\sqrt{3})\frac{R_n}{B}}.$$

The definite integral of which is:

$$\int_{d_n}^{d_{n+1}} \frac{dx}{\left[x-(3+\sqrt{3})\frac{R_n}{B}\right]\left[x-(3-\sqrt{3})\frac{R_n}{B}\right]}$$

$$=+\frac{\sqrt{3}B}{6R_n}\log_\epsilon \frac{\left[d_{n+1}-(3+\sqrt{3})\frac{R_n}{B}\right]\left[d_n-(3-\sqrt{3})\frac{R_n}{B}\right]}{\left[d_{n+1}-(3-\sqrt{3})\frac{R_n}{B}\right]\left[d_n-(3+\sqrt{3})\frac{R_n}{B}\right]}. \quad (l)$$

The integration of Eq. (h) may now readily be accomplished by simply substituting in that equation the right-hand members of Eqs. (i), (j), (k), and (l), above, for their corresponding left-hand members, as they occur in Eq. (h).

By performing the indicated operations, collecting terms, and writing, in place of the Napierian logarithm, the conversion factor, 2.30259, so that common logarithms may be used in computations, there results Eq. (13), for $E\Delta_n$, as finally written on page 153. (Note that l_n has been substituted for Bd_n and l_{n+1} for Bd_{n+1}, in this expression.)

Derivation of $E\theta_n$.

Eq. (10), of page 150, is:

$$d\theta=\frac{Mdx}{EI}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

The general integral of which may be written:

$$E\int d\theta=\int\frac{M}{I}dx. \quad . \quad . \quad . \quad . \quad . \quad . \quad (m)$$

Substituting (f'), $\frac{1}{I}$, from page 211 and the expression for M_n, Eq. (11), in the above Eq. (m), and collecting terms, gives:

$$E\int_{d_n}^{d_{n+1}} d\theta_n = \frac{36R_n}{B^4}\int_{d_n}^{d_{n+1}}$$

$$\left\{ -\frac{\frac{2R_n}{B}(P_1d_1+P_2d_2+\ldots+P_nd_n)dx}{x^3\left[x-(3+\sqrt{3})\frac{R_n}{B}\right]\left[x-(3-\sqrt{3})\frac{R_n}{B}\right]} \right.$$

$$+\frac{\frac{2R_n}{B}(P_1+P_2+\ldots+P_n)+(P_1d_1+P_2d_2+\ldots+P_nd_n)}{x^2\left[x-(3+\sqrt{3})\frac{R_n}{B}\right]\left[x-(3-\sqrt{3})\frac{R_n}{B}\right]}dx$$

$$\left. -\frac{(P_1+P_2+\ldots+P_n)dx}{x\left[x-(3+\sqrt{3})\frac{R_n}{B}\right]\left[x-(3-\sqrt{3})\frac{R_n}{B}\right]} \right\} \quad \ldots\ldots \quad (n)$$

Obviously, the definite integrals of Eqs. (i), (j), and (k), as developed, are directly applicable to Eq. (n), above.

Substituting the right-hand members of Eqs. (i), (j), and (k), for their corresponding left-hand members, as they occur, in Eq. (n), performing the indicated operations, collecting terms, together with substitution of the conversion factor, 2.30259, and l_{n+1} and l_n, as explained in the treatment of Eq. (h), above, will result in Eq. (14), for $E\theta_n$, as finally written on page 153.

Deflection equations, for Arched Dam of rectangular, vertical cross-section.

Derivation of $EI\Delta_n$.

The moment of inertia, being a constant, becomes a factor with E on the left side of the equation.

As before, Eq. (9) applies, or, as $d_n = na$,

$$d\Delta = \frac{M(x-na)dx}{EI}. \quad . \quad . \quad . \quad . \quad . \quad (o)$$

Whence

$$EI\int_{na}^{(n+1)a} d\Delta_n = \int_{na}^{(n+1)a} M(x-na)dx. \quad . \quad . \quad (p)$$

Substituting the value for M_n from Eq. (15) of page 167 and performing the integrations indicated and collecting terms, Eq. (16) results directly.

Derivation of $EI\theta_n$.

Eq. (17) of page 167 may be similarly derived by means of Eq. (10), which may be written

$$EI\int_{na}^{(n+1)a} d\theta_n = \int_{na}^{(n+1)a} Mdx,$$

into which the value of M_n, from Eq. (15), is substituted and integrations performed, terms collected, etc., as before.

APPENDIX II

MOVEMENTS AND STRESSES IN AN ARCH SUBJECTED TO A UNIFORM, RADIAL LOAD

With Derivation of Eq. (8), of page 149, for Arch Crown Deflection.

The nomenclature of page 145, together with designations of Fig. 47 and such other as may be immediately pertinent, will obtain in the following discussion which is largely adapted from a discussion by the late R. Shirreffs.*

In Fig. 47, line $1-2'-3'$ represents one-half of the axis of a segmental arch ring in its unloaded position. For convenience of reference the line $3'-O'$ may be assumed as vertical and passing through the crown of the arch. The abutment, or skewback supporting the arch, may be assumed to be in the line $1-4-O'$. ϕ_n, then, is one-half the central angle of the arch span, and r_n the radius of the axis.

In elucidating the various analytic expressions for effects of loading both as to stressing and deflecting the arch, certain changes in position are imagined.

These are, in order, as follows:

Beginning with the unloaded arch, one-half of which is shown in Fig. 47, and assuming it either to be a portion of a closed ring or to rest upon frictionless abutments, a radial loading of intensity q_n, will produce a shortening of the

* Trans. Am. Soc. C.E., Vol. LIII, p. 163.

original length and a reduction of the radius length by a proportionate amount, f'. The half arch will be moved into a new position 4–5–6. It will be in equilibrium under only the axial thrust $q_n r_n$.

Secondly, assume the arch severed at the crown and the right half, removed, replaced by a crown thrust $= q_n r_n$

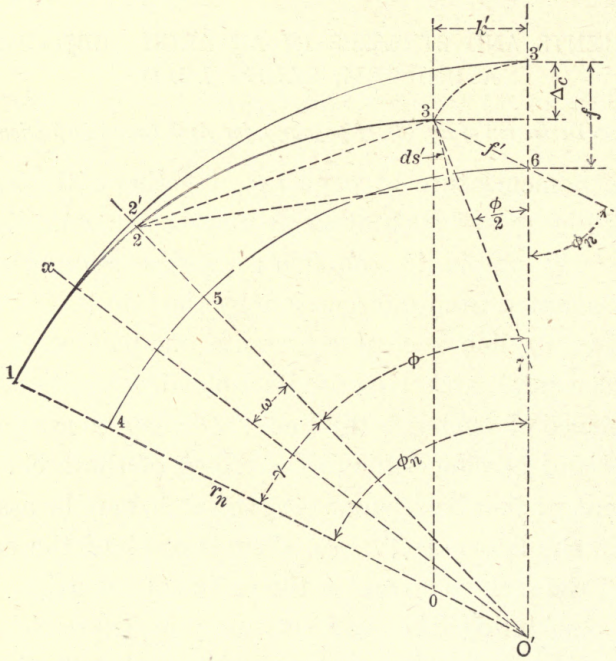


FIG. 47.

and the remaining half arch moved outward until Point 4 again coincides with Point 1. The perpendicularity of the axis at the skewback remains unchanged.

The center of the arch will be at o instead of O'; Point 6 will be in the position of Point 3, at the horizontal distance k' from the vertical through the middle of the span or crown of the arch, and at the vertical distance Δ_c below

the original crown of the arch. The crown joint will still be vertical.

Thirdly, in order to restore the integrity of the arch, under its loading, the crown thrust, $q_n r_n$, must be so diminished that under the combined action of this diminished thrust and the loads on the half arch, the curved beam 1–2–3, now considered fixed at the abutment, shall be deflected through the horizontal distance k' and as the original crown thrust, $q_n r_n$, just holds the arch in equilibrium against the action of the loads, a force H' applied at the crown and equal to the necessary diminution of $q_n r_n$, H' acting therefore to the right, will cause a movement identical with that through k'.

Fourthly, the crown joint, which will have been deflected through an angle β, by this movement, must again be made vertical in its new position. This can be accomplished only by the application of a moment, M_c.

Fifthly, the total movement of the arch at any point will be obtained by combining the movement resulting from axial stress with those movements produced by the force H' and the moment M_c.

The above considerations will next be treated analytically.

Derivation of expression Δ_c, under axial thrust.

Assume the arch ring to be of thickness $= l_n$ and a depth (normal to the plane of the paper in Fig. 47) of 1.

Let λ = shortening of the half length of arch shown in Fig. 47.

L = curved length of arch from abutment to crown.

l_n = area of radial vertical cross-section of arch ring.

E = modulus of elasticity.

Then

$$\frac{q_n r_n}{l_n} = \text{intensity of stress in ring.} \quad . \quad . \quad . \quad (1)$$

$$\frac{\lambda}{L} = \text{strain due to stress.} \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$\therefore \quad E = \frac{q_n r_n}{l_n} \div \frac{\lambda}{L}.$$

Whence

$$\lambda = \frac{q_n r_n L}{l_n E}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Since the ratio of the shortening of the arch to its original length is equal to the ratio of the change in length f', of the radius to its original length,

$$\frac{\lambda}{L} = \frac{f'}{r_n}.$$

Whence

$$f' = \frac{\lambda r_n}{L}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Combining Eqs. (3) and (4) gives

$$f' = \frac{q_n r_n^2}{l_n E}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4a)$$

And as $k' = f' \sin \phi_n$ (see Fig. 47),

$$k' = \frac{q_n r_n^2}{l_n E} \sin \phi_n; \quad . \quad . \quad . \quad . \quad (5)$$

and as $\Delta_c = f'(1 - \cos \phi_n)$, from Eq. (4$a$), there follows, in this connection:

$$\Delta_c = \frac{q_n r_n^2}{l_n E}(1 - \cos \phi_n). \quad . \quad . \quad . \quad . \quad . \quad (6)$$

At a distance of ϕ degrees from the crown there will obtain

$$\Delta_\phi : \Delta_c = \phi_n - \phi : \phi_n,$$

therefore

$$\Delta_\phi = \frac{q_n r_n^2}{l_n E} \frac{\phi_n - \phi}{\phi_n} (1 - \cos \phi_n). \quad . \quad . \quad (6a)$$

Derivation of expression for H', or diminution of stress $q_n r_n$.

The slight reduction in the compression of the arch ring due to diminishing $q_n r_n$ by the amount H' is neglected in the following.

The general expression for the differential deflection ds, of a beam is

$$ds = \frac{Mxdx}{EI},$$

in which x is referred to any point in the beam's axis, $\int ds$ is the deflection with respect to that point and M is the bending moment about that point. (See Point 2 of Fig. 47.)

In this case

$$M = H' r_n (1 - \cos \phi),$$

$$x = 2r_n \sin \frac{\phi}{2}, \text{ sufficiently close.}$$

$$dx = r_n d\phi,$$

$$I = \frac{l_n^3}{12}.$$

Substituting these last expressions in the general expression for ds, results in

$$ds' = \frac{24 r_n^3 H'}{E l_n^3} (1 - \cos \phi) \sin \frac{\phi}{2} d\phi.$$

But the horizontal component of ds' is dk',

$$dk' = ds' \sin \frac{\phi}{2},$$

since (Fig. 47) ds', approaching zero, sensibly coincides in direction with the line 3–7, the tangent to the arc ds'.

Therefore

$$dk' = \frac{12 r_n^3}{E l_n^3} H'(1 - \cos \phi)^2 d\phi. \quad . \quad . \quad . \quad . \quad (7)$$

Integrating Eq. (7) between the limits ϕ_n and 0 and equating the result with the right-hand member of Eq. (5), for k', gives

$$H' r_n = \frac{q_n l_n^2}{12} \frac{2 \sin \phi_r}{3\phi_n + \sin \phi_n \cos \phi_n - 4 \sin \phi_n}. \quad . \quad . \quad (8)$$

Derivation of expression for moment M_c, or the moment, the effect of which is to render the crown joint vertical, in position.

The moment M_c must cause the same angular deflection, β, in the entire beam as the force H'.

The general equation for angular deflection is

$$d\beta = \frac{ds}{x} = M \frac{dx}{EI}.$$

x and M are again taken with reference to the same point, which may be any point at ϕ, r_n in beam. (See Point 2.)

From the foregoing considerations, substitute for

$$M = M_c = H' r_n (1 - \cos \phi),$$

$$dx = r_n d\phi,$$

$$I = \frac{l_n^3}{12}$$

in the above expression for $d\beta$, whence

$$d\beta = \frac{12H'r_n^2}{El_n^3}(1 - \cos\phi)d\phi = \frac{12M_c r_n}{El_n^3}d\phi.$$

Integrating both expressions for $d\beta$ between the limits of $\phi = \phi_n$ and $\phi = 0$ and equating the results give:

$$M_c = H'r_n\frac{\phi_n - \sin\phi_n}{\phi_n}. \quad . \quad . \quad . \quad . \qquad (9)$$

Substituting the value of $H'r_n$ from Eq. (8) above in Eq. (9) gives for M_c,

$$M_c = \frac{q_n l_n^2}{12}\left(\frac{\phi_n - \sin\phi_n}{\phi_n}\right)\frac{2\sin\phi_n}{3\phi_n + \sin\phi_n\cos\phi_n - 4\sin\phi_n}. \quad . \quad . \qquad (10)$$

M_c is opposite in effect as to resulting deflection, to that produced by H'.

The deflection of the arch under the combined action of H' and M_c.

The object of this phase of the analysis is to determine the combined movement in a radial direction, due to H' and M_c, for this radial deflection is the arch deflection which, in a curved dam will produce stresses in the vertical, or cantilever beams.

The general expression for the deflection of a beam again serves, i.e.,

$$ds = Mx\frac{dx}{EI}.$$

The reference point or origin now will be some other point than 2, of Fig. 47, such as x, at any angle ω with the radius through Point 2.

The two moments considered with respect to Point x will be

$$H'r_n[1-\cos(\phi+\omega)] \text{ and } M_c.$$

The resulting moment, etc., will therefore be

$$M=H'r_n[1-\cos(\phi+\omega)]-M_c.$$

$$x=2r_n \sin\frac{\omega}{2}.$$

$$dx=r_n d\omega \quad \text{and} \quad I=\frac{l_n^3}{12}.$$

Substituting these in the general expression for ds above gives

$$ds'_\phi=\frac{24r_n^2}{El_n^3}\{H'r_n[1-\cos(\phi+\omega)]-M_c\}\sin\frac{\omega}{2}d\omega.$$

Let the radial component of this movement ds'_ϕ be represented by dD'_ϕ.

Then

$$dD'_\phi=ds'_\phi\cos\frac{\omega}{2},$$

therefore

$$dD'_\phi=\frac{12r_n^2}{El_n^3}\{H'r_n[1-\cos(\phi+\omega)]-M_c\}\sin\omega d\omega.$$

Integrating this last expression for dD'_ϕ, between the limits $\omega=(\phi_n-\phi)=\gamma$ and $\omega=0$ and substituting $H'r_n$ and M_c, as written in Eqs. (8) and (10) respectively, gives for D'_ϕ,

$$D'_\phi=C\frac{q_n r_n^2}{El_n}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

in which C has the general value given in Eq. (11*a*), or

$$C=\left[\frac{2\sin\phi_n}{\phi_n}(1-\cos\gamma)+\frac{\sin\phi}{2}(2\gamma-\sin 2\gamma)\right.$$
$$\left.+\frac{\cos\phi}{2}(\cos 2\gamma-1)\right]\div\left(\frac{3\phi_n}{\sin\phi_n}+\cos\phi_n-4.\right) \quad . \quad (11a)$$

When $\phi=0$ and therefore $\gamma=\phi_n$, the value of C is as given in Eq. (11*b*), or

$$C_c=\frac{\dfrac{2\sin\phi_n}{\phi_n}(1-\cos\phi_n)+\frac{1}{2}(\cos 2\phi_n-1)}{\dfrac{3\phi_n}{\sin\phi_n}+\cos\phi_n-4}, \quad . \quad . \quad (11b)$$

and for the semicircle, since $\phi_n=\frac{\pi}{2}$,

$$C_c=\frac{\dfrac{4}{\pi}-1}{\dfrac{3\pi}{2}-4} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11c)$$

To combine the effect of the axial thrust or Δ_c with the crown deflection as given in Eq. (11*b*) and Eq. (11), there should be noted the following:

The last term in the numerator of Eq. (11*b*) involves $2\phi_n$. This may be written in terms of ϕ_n, for

$$\tfrac{1}{2}(\cos 2\phi_n-1)=(\cos^2\phi_n-1),$$

also, the coefficient of C in Eq. (11), $\frac{q_n r_n^2}{El_n}$ is the same as the coefficient of $(1-\cos\phi_n)$ in Eq. (6) for Δ_c. Hence the combined trigonometric function of ϕ_n designated as CC_c, Fig. 28, which is the coefficient plotted as a curve, referred to before, for the common factor of Eqs. (11) and (6),

viz., $\frac{q_n r_n^2}{El_n}$. This combination is written as Eq. (8) on page 149 for D_c.

Derivation of the moment M_ϕ, at any point, ϕ degrees from the crown.

The moment is $H'r_n(1-\cos\phi)-M_c$.

Substituting the values of $H'r_n$ and M_c as written in Eqs. (8) and (10), and reducing give

$$M_\phi=\frac{q_n l_n^2}{6}\,\frac{\dfrac{\sin\phi_n}{\phi_n}-\cos\phi}{\dfrac{3\phi_n}{\sin\phi_n}+\cos\phi_n-4}. \quad . \quad . \quad . \quad (12)$$

M_ϕ becomes 0, or there is a point of contra flexure in the curved beam used as an arch under a uniform radial load, when

$$\cos\phi=\frac{\sin\phi_n}{\phi_n},$$

or in the semicircle when

$$\cos\phi=\frac{2}{\pi},$$

or at about 50° from the crown.

The stresses of tension or compression produced by M_ϕ at any point in the arch may be combined with the axial stress $q_n r_n$, whence the resultant stress at that point may be obtained. The force H', except in very flat arches, may usually be neglected in getting the resultant stresses.

APPENDIX III

CROSS-SECTIONS OF EXISTING MASONRY DAMS

THIS selection of cross-sections of notable masonry dams, following Table XIV, is arranged in chronological order to illustrate the evolution of the masonry dam, the design of which has been largely a matter of following precedent, in many cases.

The series begins with a few of the heavy, Spanish type and continues through the French designs, such as those of de Sazilly and Delocre, to the types of present-day construction, as illustrated by foreign and American (United States) examples.

With respect to the design of de Sazilly, reference to whom was made early in Chapter I, it should be stated that his analysis resulted in a cross-section that was necessarily stepped, and Delocre suggested substituting curved faces in place of the steps, thereby effecting an appreciable saving of material in construction.

Attention is directed to the fact that all of the sixty-one sections could not be shown to the same scale; but each cross-section is sufficiently dimensioned so that comparisons are possible.

The various methods of designating batters are also apparent.

The later, overfall types only, have been grouped separately.

Table XIV contains a page index of the cross-sections, together with other data concerning each, for convenience of reference.

In the table, under "Type of Section,"

G = gravity section (depending on gravity);
A = arch section (depending on arch action);

and under "Type of Service,"

N.O. = non-overflow;
S. = spillway dam.

A dash, in the column, headed "Radius, etc.," denotes a straight alignment of dam.

TABLE XIV

Masonry Dams

Name.	Cross-Section on Page	Type of		Radius if Arched in Plan. (Feet.)	Year		Max. Height (Feet.)	Designers or Builders.	Location.	Remarks.
		Section.	Service.		Begun.	Finished.				
Almanza.....	236	G.	N. O.	86	Prior to	1586	68	Spanish.....	Spain.....	Oldest existing masonry dam Upper part.* Lower 48 ft. on curve. Rubble and cut stone.
Arrowrock.....	261	G.	N. O.	662	1912	1915	321	American...	Idaho.....	Boise River. United States Reclamation Service.
Assuan.....	254	G.	N. O.	—	1898	1903	115;131	English.....	Egypt.....	Sluiceways through dam. Raised to 131 feet in 1909.
Ban.....	240	G.	N. O.	1325	1866	1870	157	French.....	France.....	St. Chamond Water Supply. Rubble masonry.
Barossa.....	254	A.	N. O.	200	1899	1903	113	English.....	Australia.....	Cyclopean masonry. Construction, temp. range, 30°–168° F.
Beetaloo.....	247	G.	N. O.	1414	1886	1889	110	English.....	Australia.....	Concrete masonry.
Betwa.....	264	G.	S.	—	About	1888	66	English.....	India.....	Alignment irregular to follow ledge. Rubble in hydraulic lime.
Big Bend.....	266	G.	S.	340	1909	.....	160	American...	California.....	Cyclopean masonry. (Designed by methods of Chapters VI and VII.)
Bouzey.....	243	G.	N. O.	—	1879	1880	67	French.....	France.....	Slipped one foot in 1884 when reservoir filled.
Bouzey.....	243	G.	N. O.	—	.....	1889	67	French.....	France.....	This section shows reinforcement in 1889. Overturned, 1895.
Boyds Corners..	241	G.	N. O.	—	1866	1873	78	American...	New York....	New York City Water Supply.
Bridgeport.....	246	G.	N. O.	—	1886	1887	45	American...	Connecticut...	Rubble in Rosendale cement mortar.
Chartrain.....	249	G.	N. O.	1312	1888	1892	175	French.....	France.....	Known also as the Tâche Dam. Rubble.
Colorado.....	265	G.	S.	—	1891	1892	66	American...	Texas.....	Near Austin. Failed by sliding, 11 ft. head on crest. Rubble.
Cross River....	258	G.	N. O.	—	1905	1909	155	American...	New York.....	N. Y. City Water-supply. Cyclopean masonry; concrete face blocks.

* Polygonal in Plan.

TABLE XIV—*Continued*

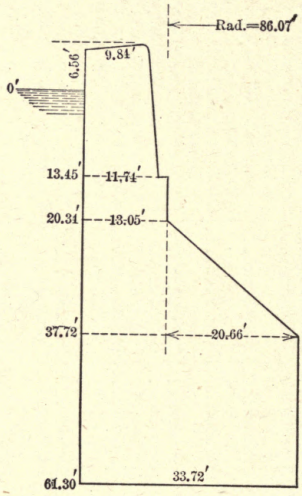
Name.	Cross-Section on Page	Type of		Radius if Arched in Plan. (Feet).	Year		Max. Height (Feet).	Designers or Builders.	Location.	Remarks.
		Section.	Service.		Begun.	Finished.				
Croton Falls...	259	G.	N. O.	—	1906	1911	167	American...	New York....	N. Y. City Water-supply. Cyclop. masonry; concrete face blocks.
Einsiedel.....	251	G.	N. O.	1310	1890	1894	92	German....	Germany.....	Chemnitz Water Supply. Rubble masonry.
Elephant Butte	262	G.	N. O.	—	1912	†	264‡	American...	New Mexico..	U. S. Reclamation Service.
Eschbach.....	250	G.	N. O.	410	1889	1891	82	German....	Germany.....	Remscheid Water Supply.
Farnham.....	263, 267	G.	N. O., S	—	.....	1912	115	American...	Massachusetts	Near Pittsfield. Expansion joints similar to those of Kensico Dam.
Furens.....	239	G.	N. O.	828	1862	1866	171	French.....	France.....	Near St. Etienne. Delocre design. Is watertight dam.
Gileppe.....	241	G.	N. O.	1640	1870	1875	157	Belgian.....	Belgium.....	Rubble and cut stone.
Gran Cheurfas.	245	G.	N. O.	—	1882	1884	131	French.....	Algiers.....	Partially failed in 1885. Rubble.
Gros Bois.....	237	G.	N. O.	—	1830	1838	93	French.....	France.....	Foundation, soft rock. Buttressed, and in 1896 earth filled on face.
Habra.....	240	G.	N. O.	—	1865	1871	125	French.....	Algiers.....	Failed in 1881, flood 5 ft. over crest. Repaired in 1883–7. New profile not found.
Hamiz.....	244	G.	N. O.	—	1880	1885	125	French.....	Algiers.....	Failed. Rubble masonry.
Hijar.....	244	G.	N. O.	210	.....	1880–87	141	Spanish.....	Spain.....	Almeria. Two dams. Rubble with cut stone facing.
Indian River...	253	G.	N. O.	—	.....	1898	47	American...	New York....	At foot of Indian Lake, N. Y.
Kensico.....	260	G.	N. O.	—	1912	†	290	American...	New York....	Thermophones installed to detect internal temperature changes.
Komotau.....	255	G.	N. O.	820	1900	1903	119	Austrian....	Franconia.....	Tar and asphalt waterproofing in up-stream face. Drainage system.
La Grange.....	265	G.	S.	300	1890	1893	128	American...	So. California.	Rubble. Has had 15 ft. depth of water on crest.
Lake Cheesman	255	G.	N. O.	400	1900	1904	227	American...	Colorado.....	For Denver Water Supply. Granite in Portland cement mortar.
Lauchensee....	250	G.	N. O.	2950	1889	1895	98	German....	Germany.....	Cyclopean masonry.
Mercedes.....	256	G.	N. O.	197	1902	1905	120	Mexican and American..	Mexico.....	Straight for half its length. Rest arched.
Mouche.....	245	G.	N. O.	—	1885	1890	101	French.....	France.....	Three-coat, pitch waterproofing in up-stream face.
New Croton...	252	G.	N. O.	—	1892	1906	296	American...	New York....	New York City Supply. Spillway at one end.

Olive Bridge...	96	G.	N. O.	—	1907	1914	220	American...	New York....	N.Y. City Supply. (Contract Drawing.) *Built* with block parapet.
Pathfinder.....	259	A.	N. O.	150	1906	1910	210	American...	Wyoming.....	North Platte Reclam. Project, U. S.
Periyar.....	251	G.	N. O.	—	1889	1896	175	English.....	So. India.....	Hydraulic lime concrete. Rubble facing.
Pont.....	242	G.	N. O.	1312	1878	1881	88	French.....	France.....	
Puentes.....	237	G.	N. O.	*	1785	1791	164	Spanish.....	Spain.....	Lorca, Murcia. Failed in 1802 by foundation grillage washing out.
Roosevelt.....	257	G.	N. O.	410	1905	1911	260	American...	Arizona.....	U. S. Reclamation. Cyclopean masonry, rubble facing.
San Mateo....	248	G.	N. O.	632	1887	1888	170	American...	California.....	San Francisco Water Supply. Built only to 146 ft. One of the first concrete dams in U. S.
Settons.....	238	G.	N. O.	—	1855	1858	62	French.....	France.....	For navigation.
Shoshone.....	258	A.	N. O.	150	1905	1910	305	American...	Wyoming.....	Shoshone Reclamation project. Concrete.
Sodom.....	249	G.	N. O.	—	1888	1892	95	American...	New York....	New York City Water Supply. Tight dam.
Spier's Falls....	256	G.	N. O.	—	1900	1905	152	American...	New York....	Cyclopean masonry. Has an 80 foot spillway section.
Sudbury.....	266	G.	S.	—	1894	1897	76	American...	Massachusetts	Boston, Metropolitan Water Works.
Sweetwater....	246	A.	N. O.	213	1886	1888	99	American...	California.....	In 1895 was overtopped 22 ins. without damage.
Tansa.....	247	G.	N. O.	—	1886	1892	118	English.....	India.....	Near Bombay. Has 1650 ft. of waste weir. Designed for 17 ft. height increase.
Ternay.....	239	G.	N. O.	1312	1861	1867	125	French.....	France.....	Granite rubble.
Thirlmere.....	248	G.	N. O.	Curved	1886	1893	60	English.....	England.....	Reverse curve in plan to follow bed-rock.
Titicus.....	252	G.	N. O.	—	1890	1895	109	American...	New York....	N. Y. City Water Supply.
Urft River.....	257	G.	N. O.	656	1901	1904	190	German....	Prussia.....	Drains provided in body of dam.
Val de Infierno.	236	G.	N. O.	*	1785	1791	116	Spanish.....	Spain.....	Province of Murcia.
Villar.....	242	G.	N. O.	440	1870	1878	170	Spanish.....	Spain.....	Rubble in hydraulic lime mortar.
Vyrnwy.....	264	G.	S.	—	1881	1888	136	English.....	No. Wales....	Liverpool, England, Water Supply.
Wachusett.....	253	G.	N. O.	—	1896	1905	205	American...	Massachusetts	Water Supply, Boston district.
Wier—Dividing	103	G.	S.	—	1912	1914	40±	American...	New York....	Ashokan Reservoir. N. Y. City Water Supply. Bridge piers at intervals.
Weir—Waste ..	104	G.	S.	—	1910	1912	20±	American...	New York....	Ashokan Reservoir. N. Y. City Water Supply. Alignment broken.
Zola.....	238	A.	N. O.	158	.....	1843	126	French.....	Aix, France...	Stone in Portland cement mortar.

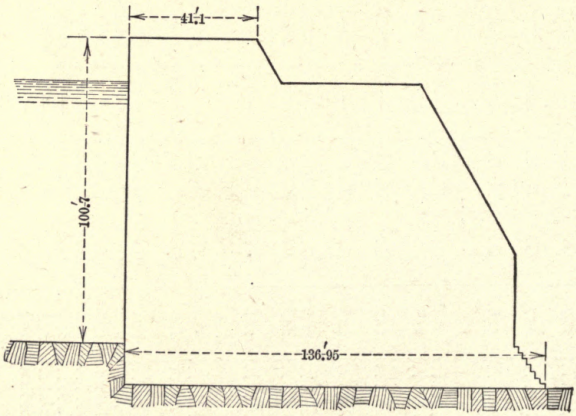
* Polygonal in Plan. † Under construction, 1916. ‡ 300± to deepest foundation.

CROSS-SECTIONS OF EXISTING DAMS

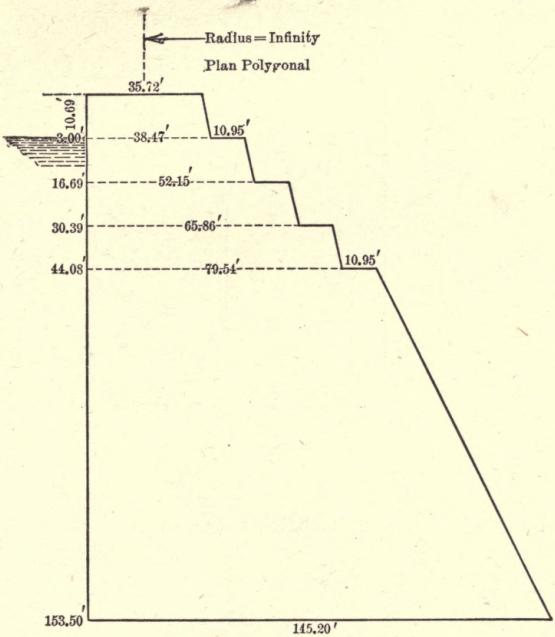
(Scales are as indicated by dimensions)



ALMANZA
Spain
Constr. prior to 1586



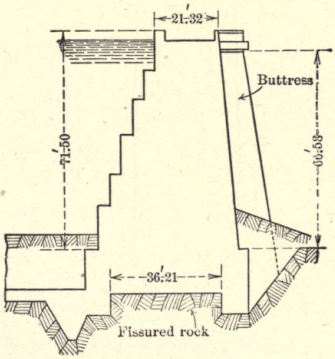
VAL DE INFIERNO
Spain
Constr. 1785–1791



PUENTES

Guadalentin River, Spain

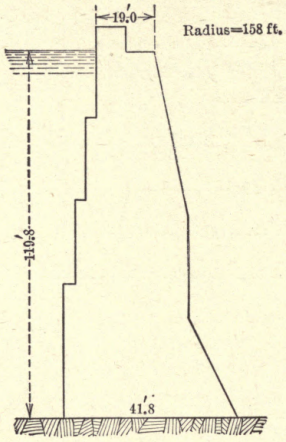
Constr. 1785–1791. Failed 1802



GROS BOIS

River Brenne, France

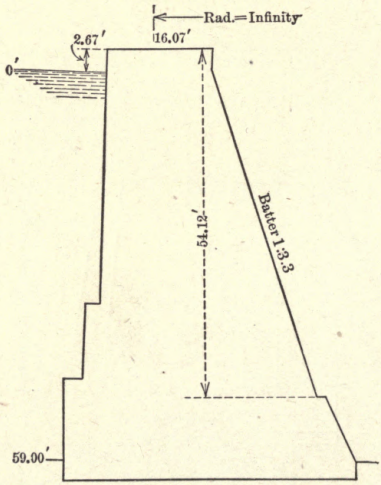
Constr. 1830–1838



ZOLA

Nea Aix, France

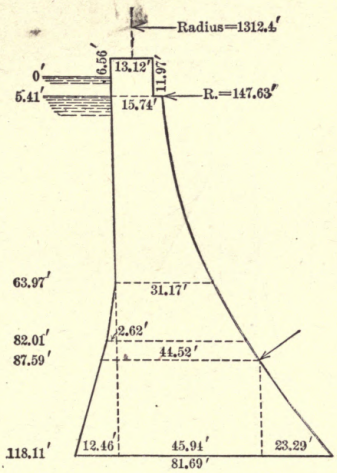
Constr., 1843



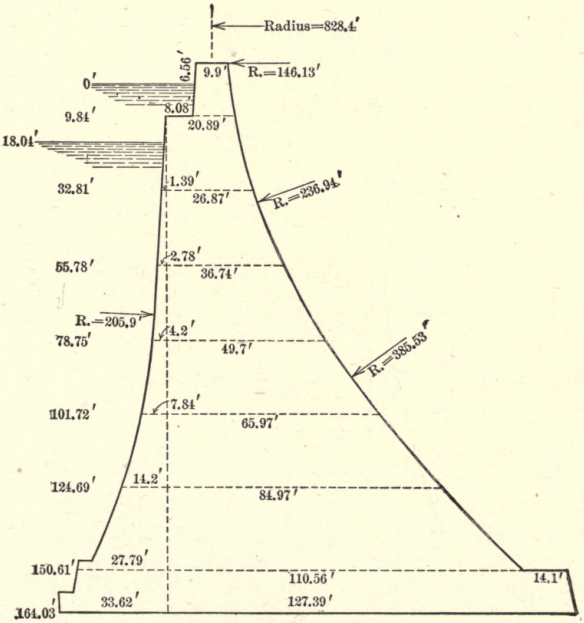
SETTONS

Yonne River, France

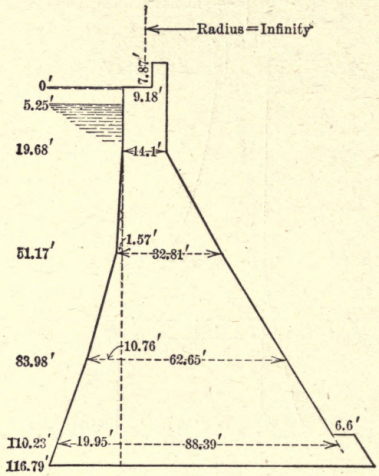
Constr. 1855–1858



TERNAY
Ternay River, France
(Bouvier design)
Constr. 1861–1867



FURENS
Furens River, France
(Delocre design)
Constr. 1862–1866



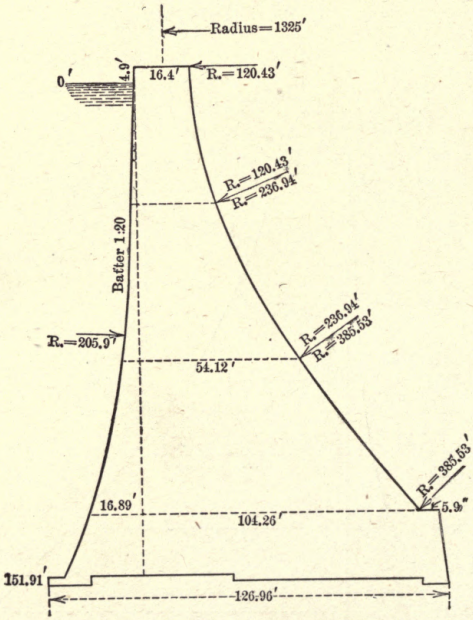
Profile of new part of Habra dam differs from old, but new profile has not been found.

HABRA

Habra River, Algiers

(Old profile)

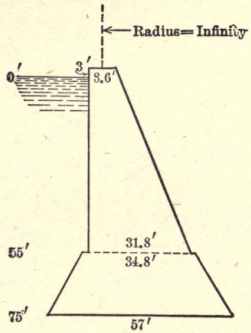
Constr. 1865–1871
Failed 1881
Repaired 1883-1887



BAN

Ban River, France

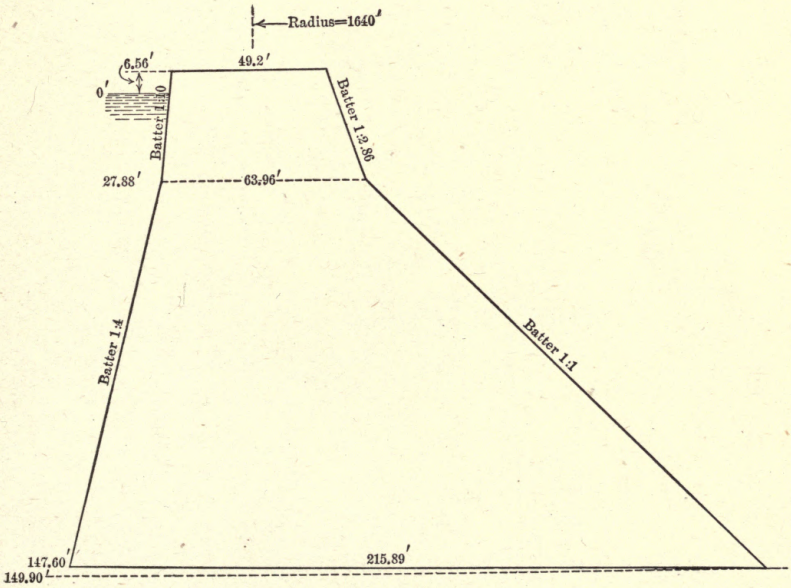
Constr. 1866–1870



BOYDS CORNERS

West Branch of Croton River, New York, U. S. A.

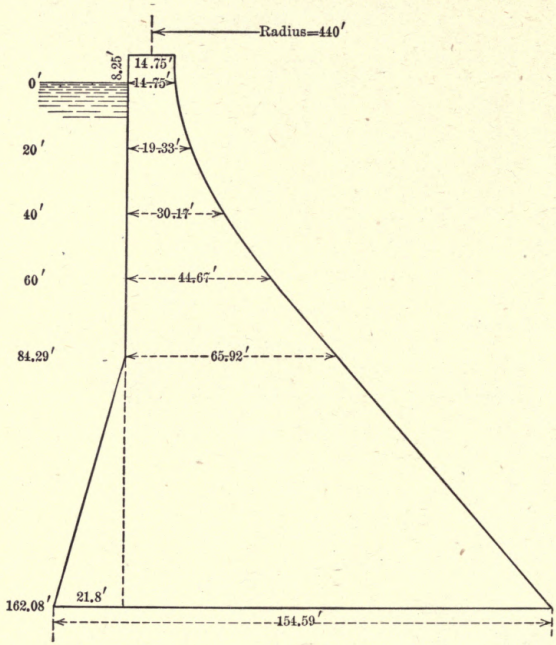
Constr. 1866–1873



GILEPPE

Belgium

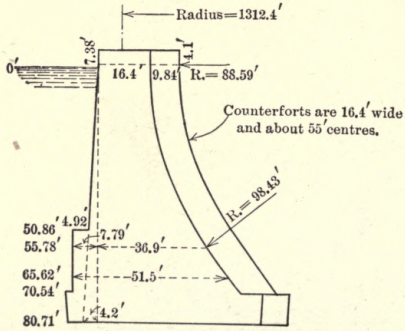
Constr. 1870–1875



VILLAR

Lozoya River, Spain

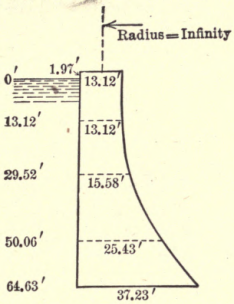
Constr. 1870–1878



PONT

Armaçon River, France

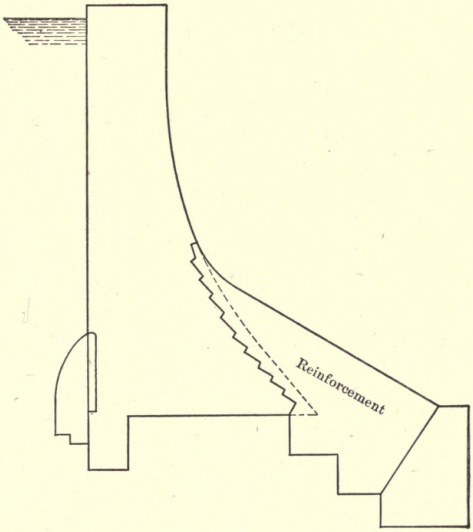
Constr. 1878–1881



BOUZEY

Avière River, France

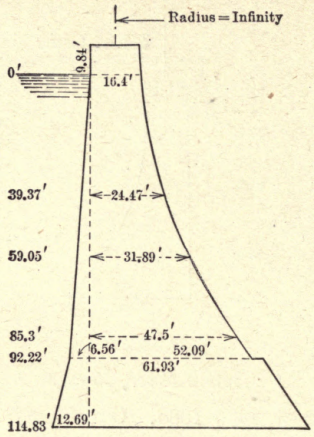
Constr. 1879–1880. Overturned, 1895



BOUZEY

Large scale profile showing reinforcement of 1888–1889

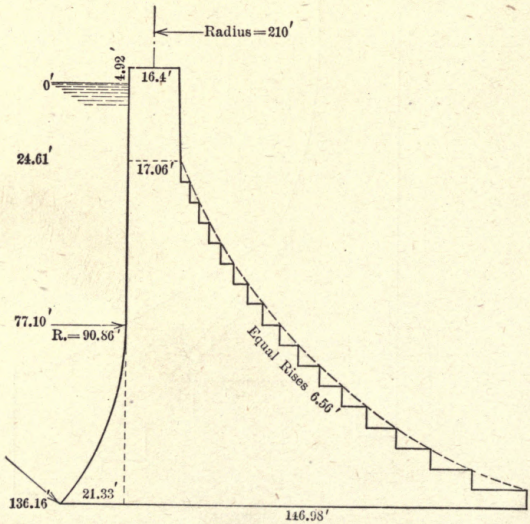
Overturned in 1895, 580 feet going out



HAMIZ

Algiers

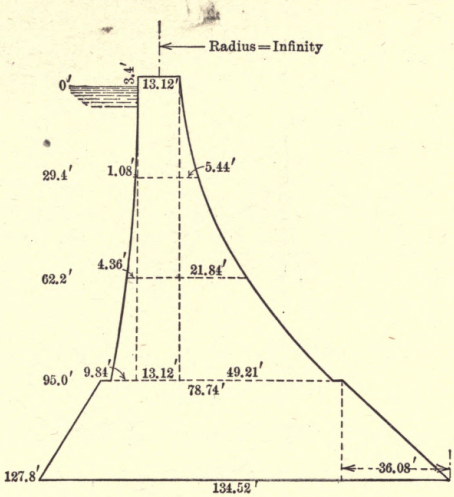
Constr. 1880–1885



HIJAR

Two such dams across the *Martin River, Spain*

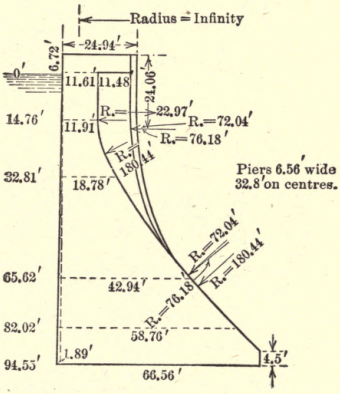
Constructed { (*a*) 1880
(*b*) 1887



GRAN CHEURFAS

Sig River, Algiers

Constr. 1882–1884

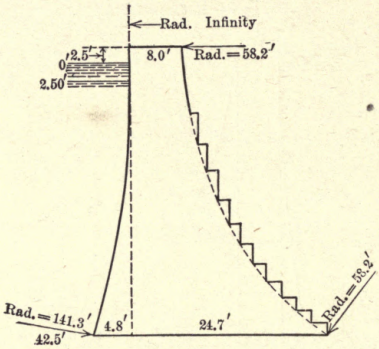


MOUCHE

Mouche River, France

Constr. 1885–1890

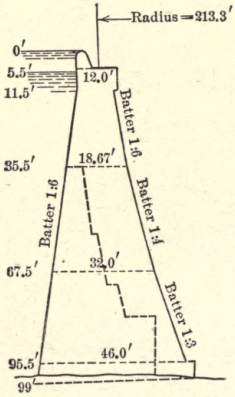
Seven temperature cracks appeared during winter following completion, and in the summer temperature change bowed the structure slightly.



BRIDGEPORT

Mill River, Connecticut, U. S. A.

Constr. 1886–1887



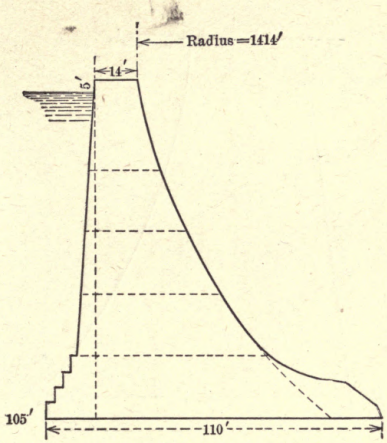
SWEETWATER

Sweetwater River, California, U. S. A.

(Schuyler design)

Constr. 1886–1888

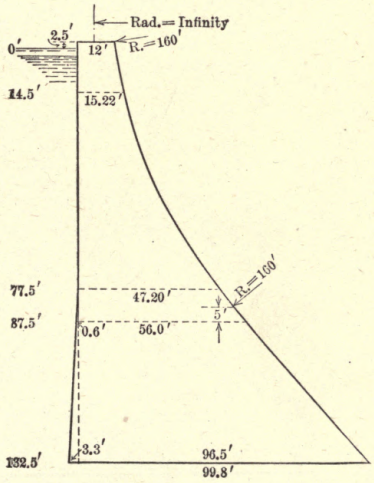
Additions, 1895



BEETALOO

Australia

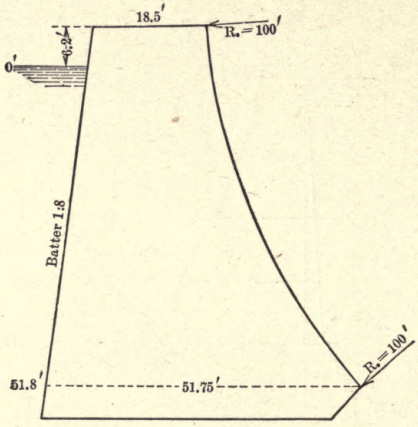
Constr. 1886–1889



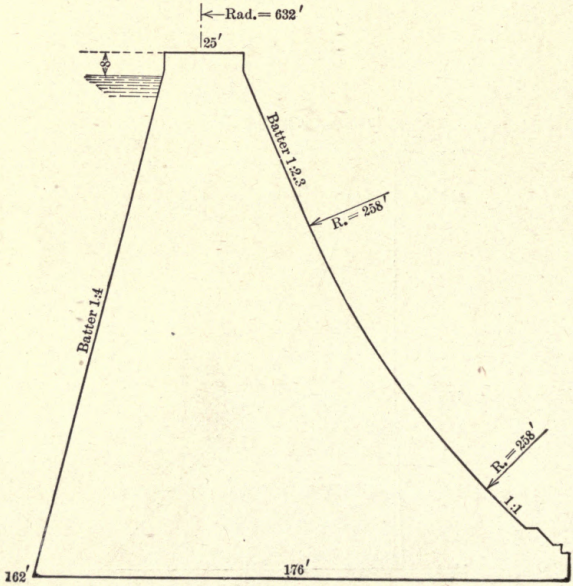
TANSA

Tansa River, India

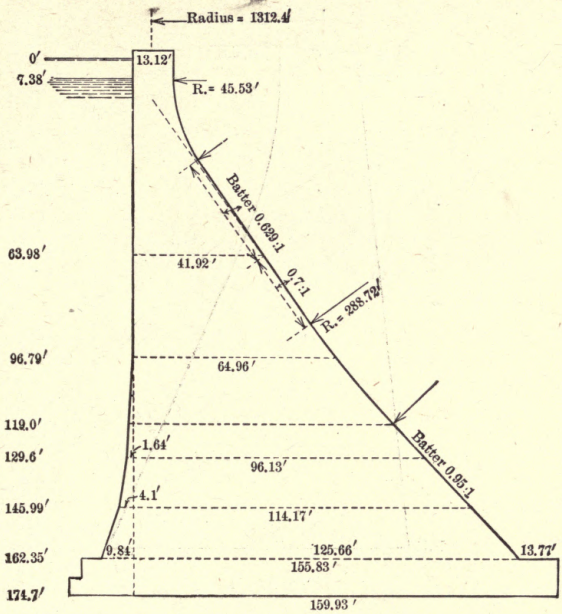
Constr. 1886–1892



THIRLMERE
England
Constr. 1886–1893



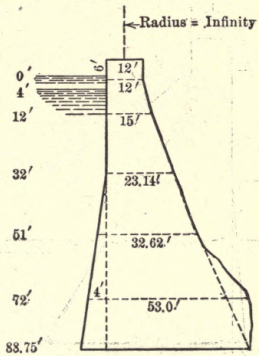
SAN MATEO
California, U. S. A.
Constr. 1887–1888



CHARTRAIN OR TÂCHE

Tâche River, France

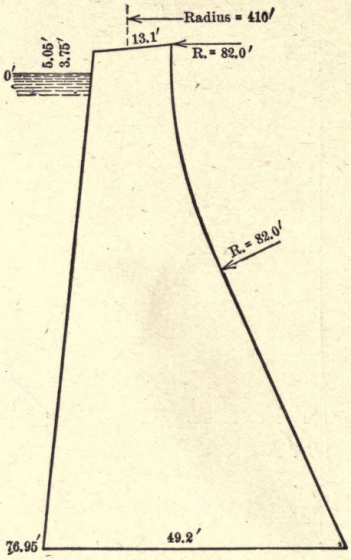
Constr. 1888–1892



SODOM

East Branch of Croton River, New York, U. S. A.

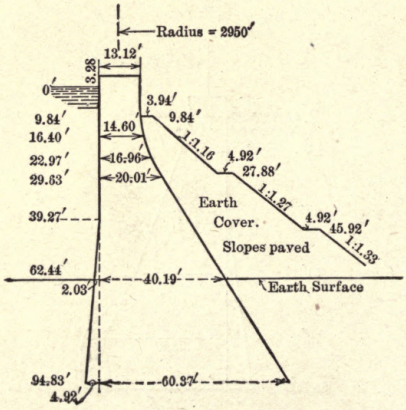
Constr. 1888–1892



ESCHBACH

Germany

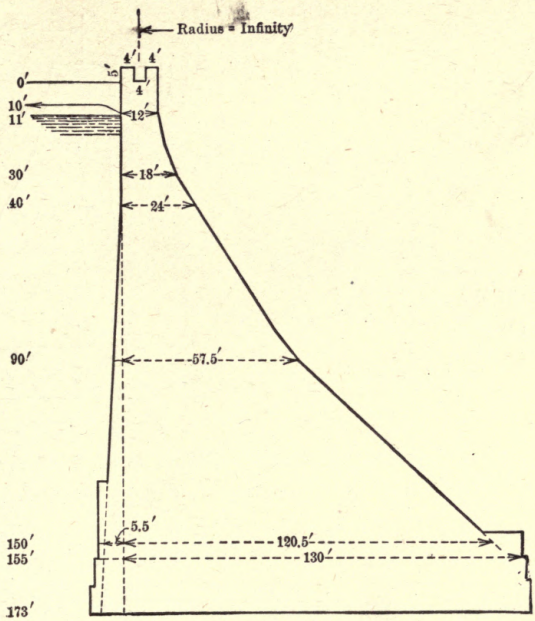
Constr. 1889–1891



LAUCHENSEE

Germany

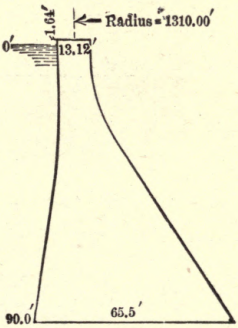
Constr. 1889–1895



PERIYAR

Periyar River, India

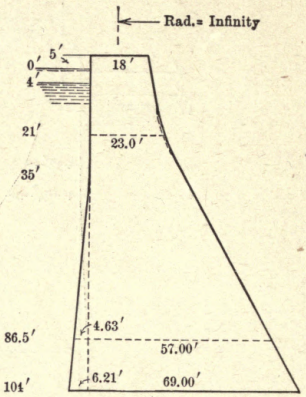
Constr. 1889–1896



EINSIEDEL

Near Chemnitz, Germany

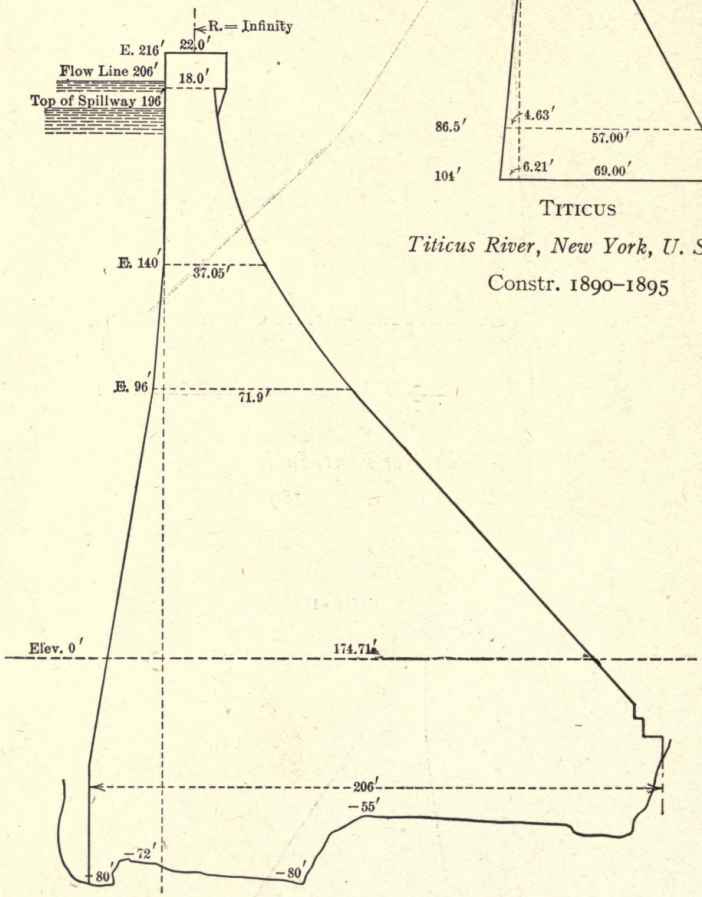
Constr. 1890–1894



TITICUS

Titicus River, New York, U. S. A.

Constr. 1890–1895

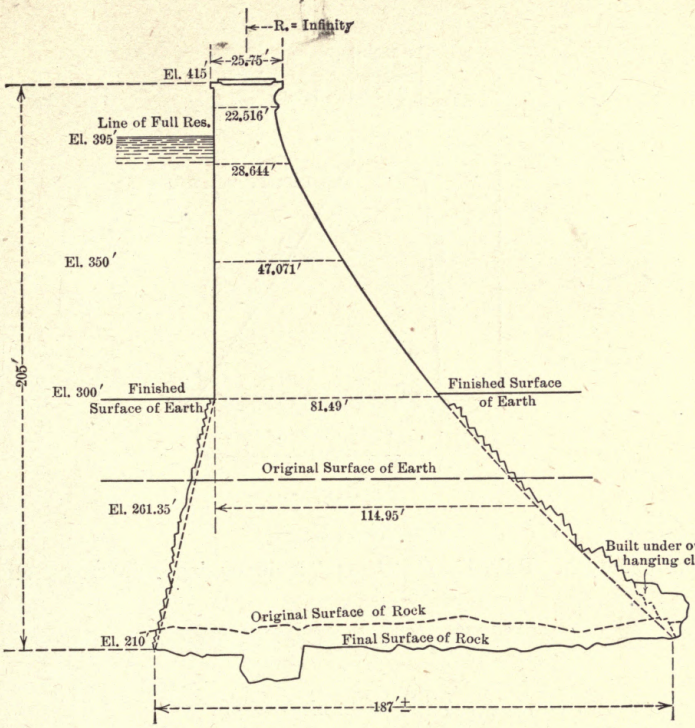


NEW CROTON

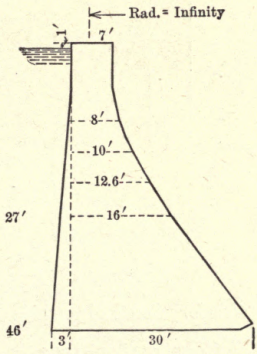
(As built)

Croton River, New York, U. S. A.

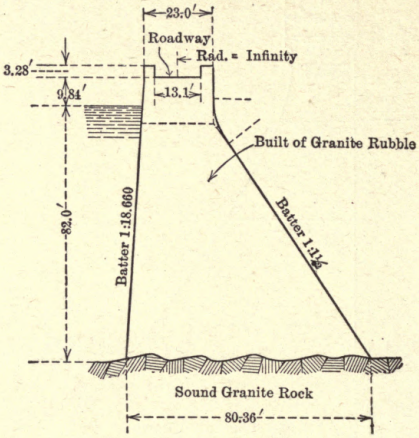
Constr. 1892–1906



WACHUSETT
Nashua River, Massachusetts, U. S. A.
Constr. 1896–1905



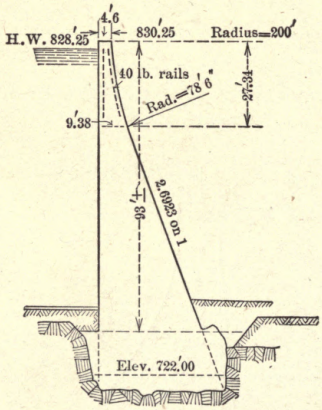
INDIAN RIVER
New York, U. S. A.
Constr. 1898



ASSUAN

Nile River, Egypt

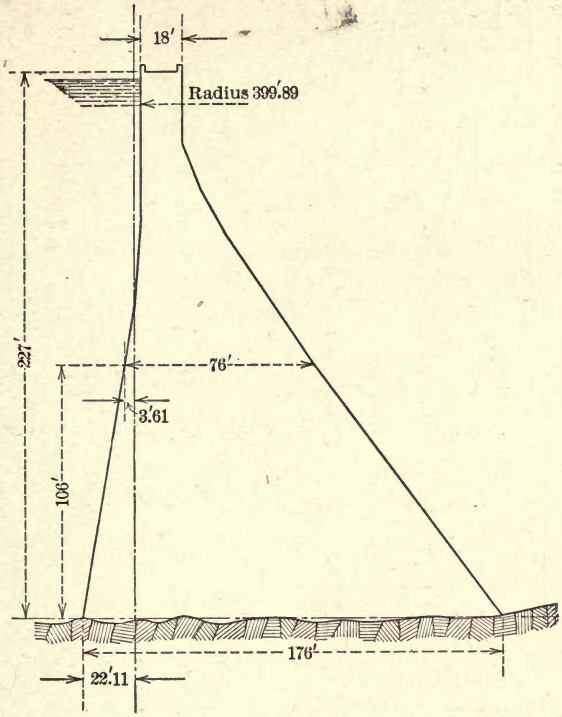
Constr. 1898–1903. Raised in 1909 to 131′, by increasing entire cross-section.



BAROSSA

Gawlor, South Australia

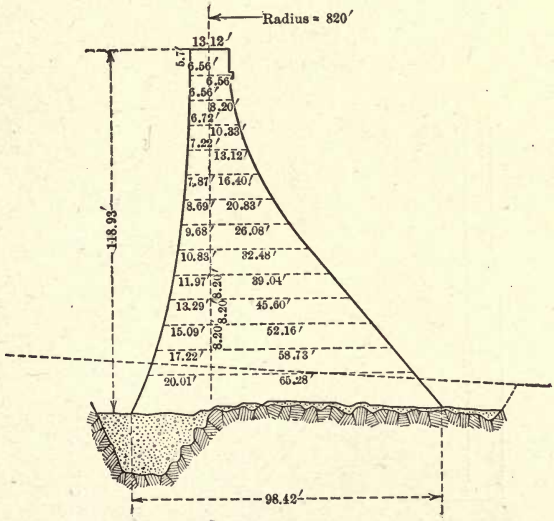
Constr. 1899–1903



LAKE CHEESMAN

Colorado, U. S. A.

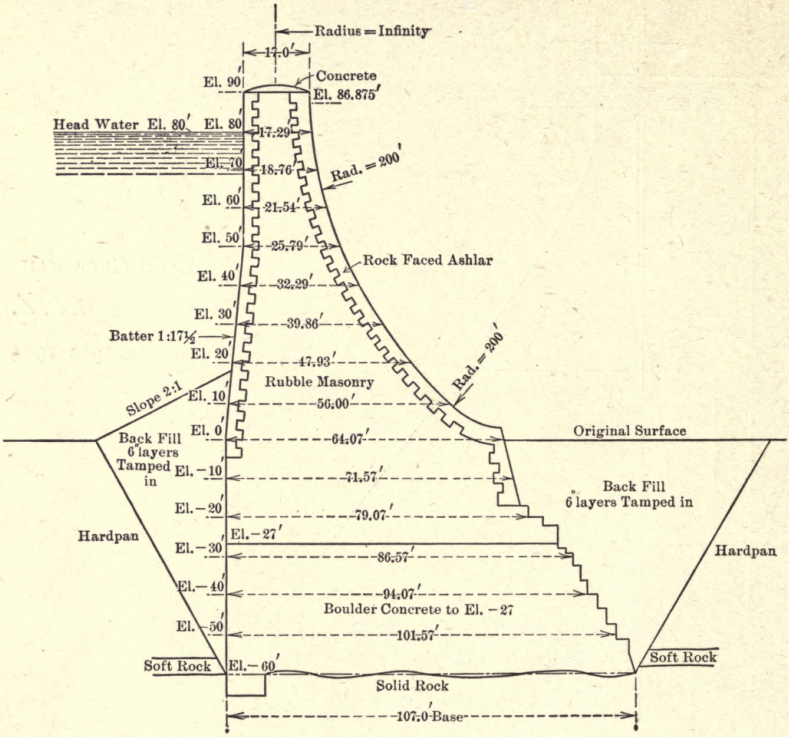
Constr. 1900–1904



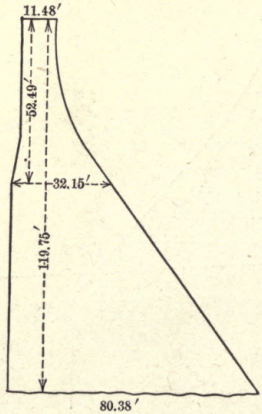
KOMOTAU

Upper Franconia, Austria

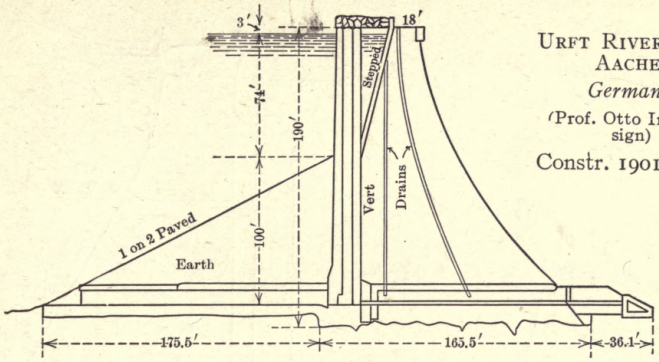
Constr. 1900–1903



SPIER'S FALLS
New York, U. S. A.
Constr. 1900–1905



MERCEDES
Durango, Mexico
Constr. 1902–1905

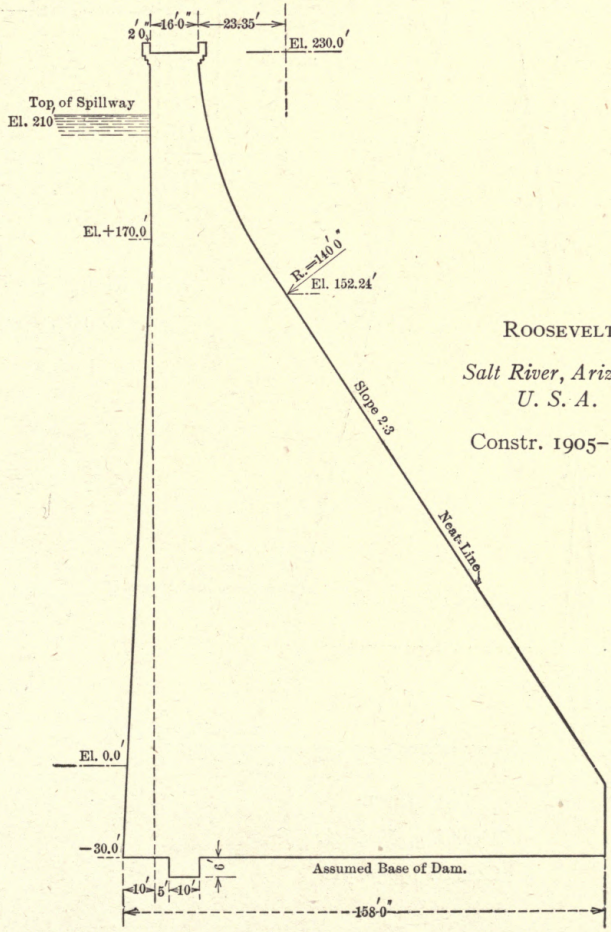


URFT RIVER, NEAR AACHEN

Germany

(Prof. Otto Intze design)

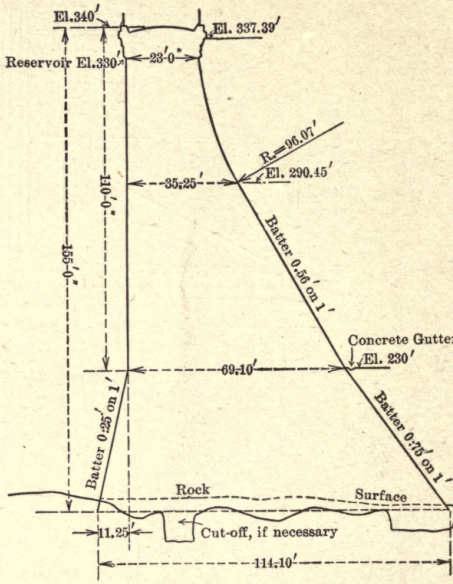
Constr. 1901–1904



ROOSEVELT

Salt River, Arizona, U. S. A.

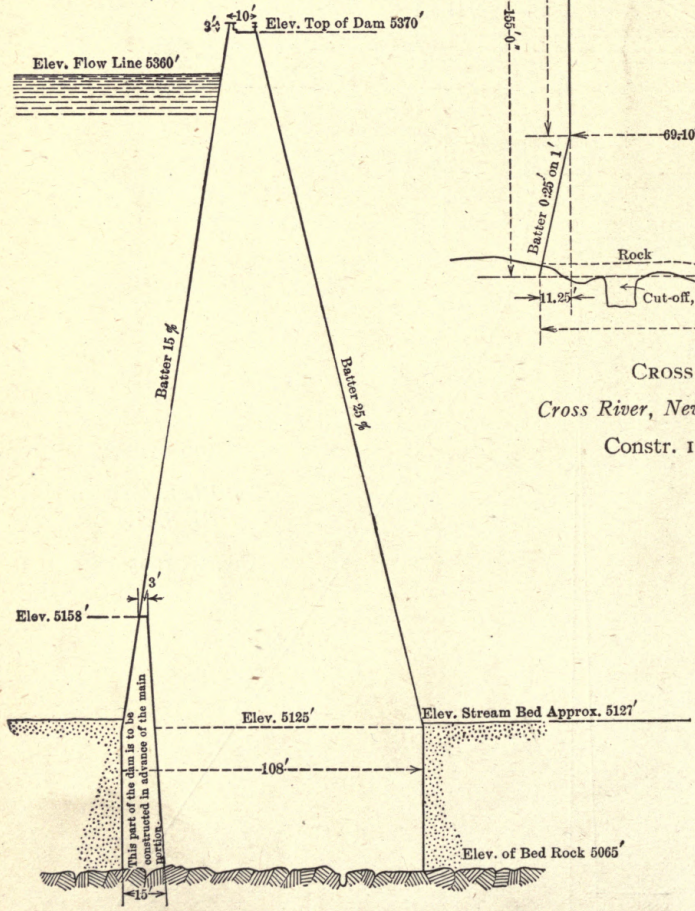
Constr. 1905–1911



CROSS RIVER

Cross River, New York, U. S. A.

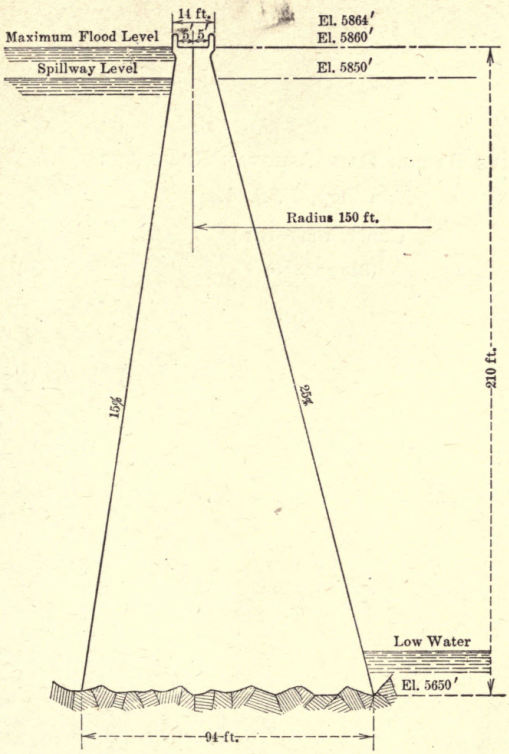
Constr. 1905–1909



SHOSHONE

Wyoming, U. S. A.

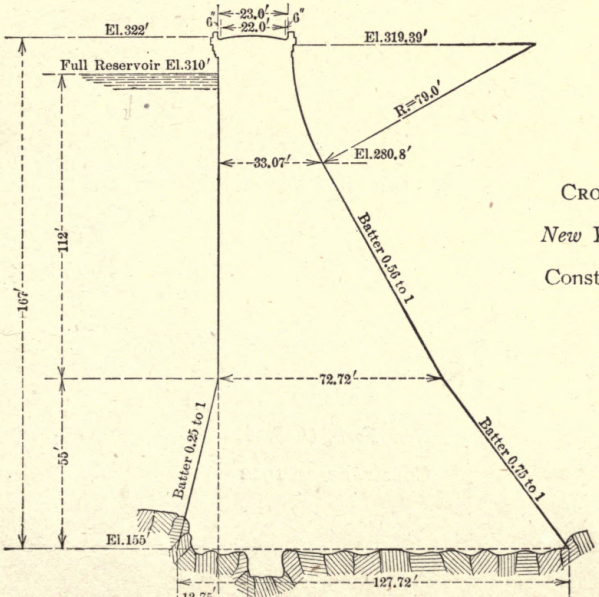
Constr. 1905–1910



PATHFINDER

North Platte River, Wyoming, U. S. A.

Constr. 1906–1910



CROTON FALLS

New York, U. S. A.

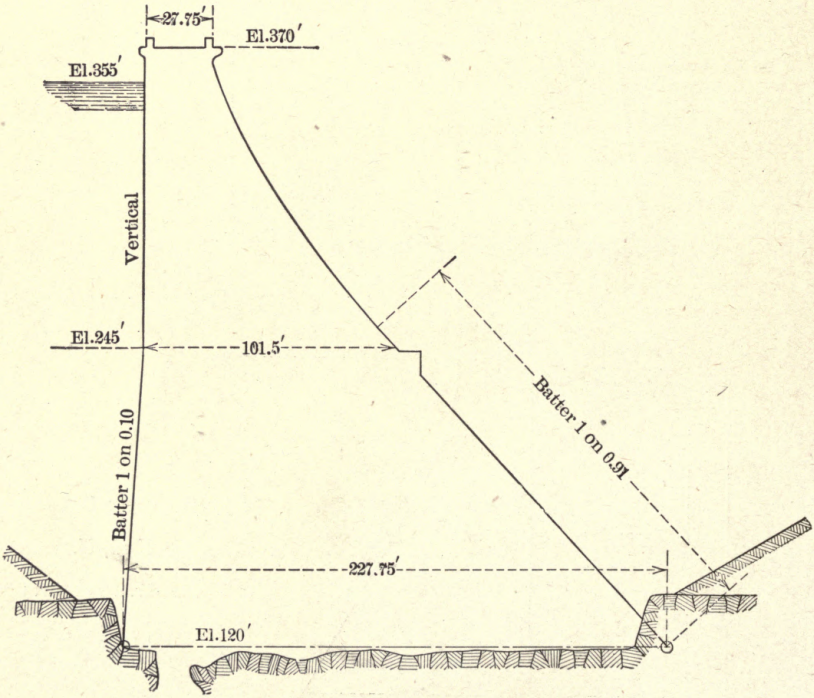
Constr. 1906–1911

OLIVE BRIDGE DAM, ASHOKAN RESERVOIR

New York, U. S. A.

Const. 1907–1914

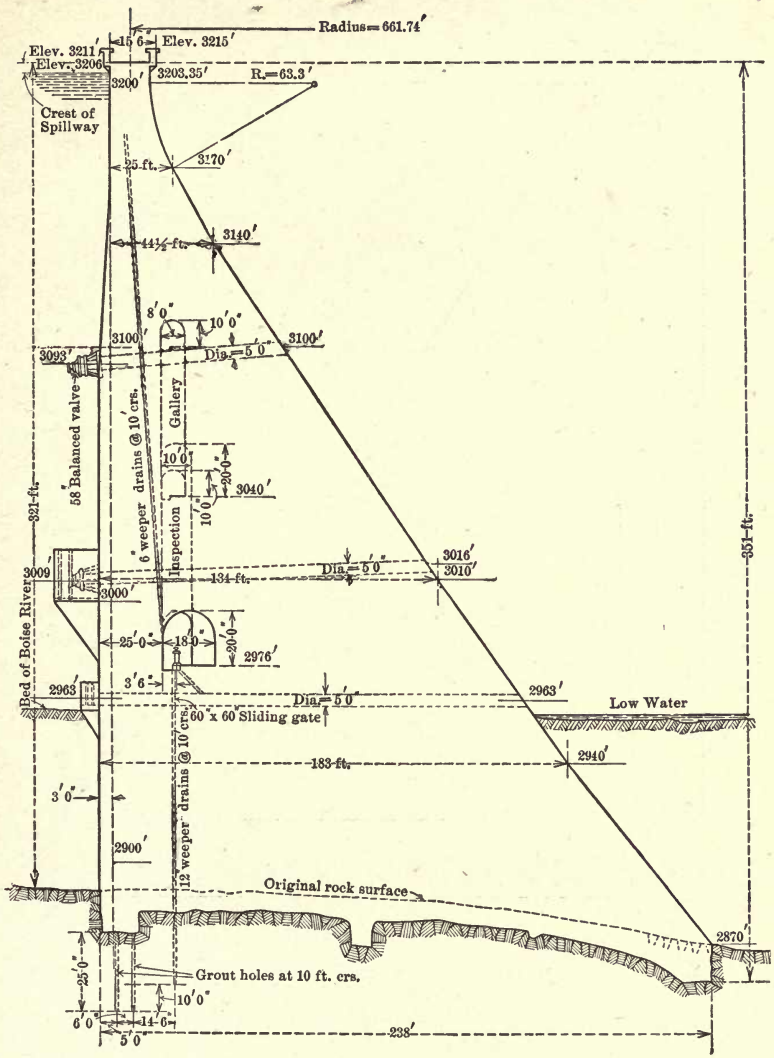
(See page 96)



KENSICO

New York, U. S. A.

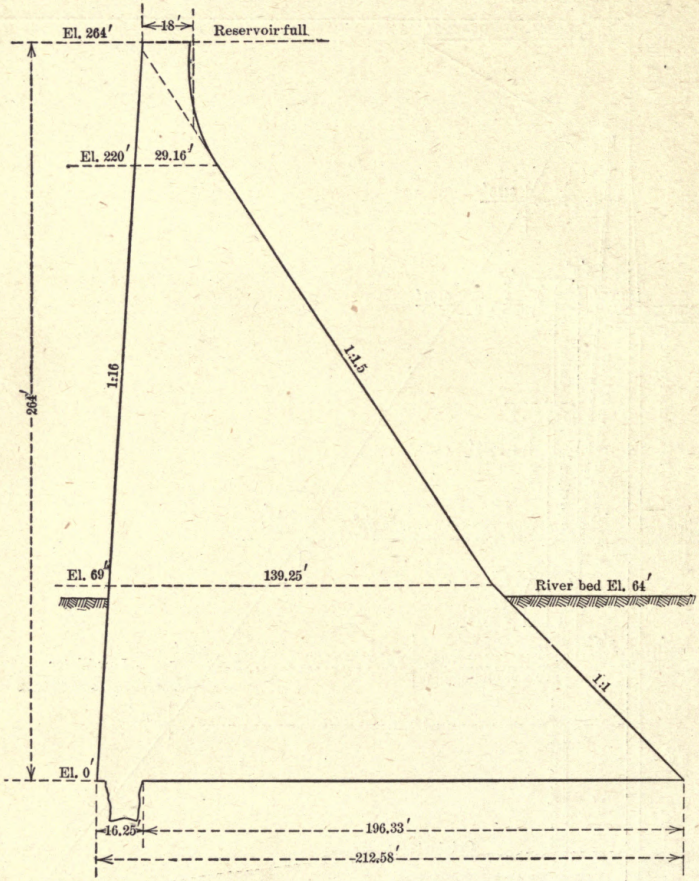
Constr. began 1912



ARROWROCK

Boise River, Idaho, U. S. A.

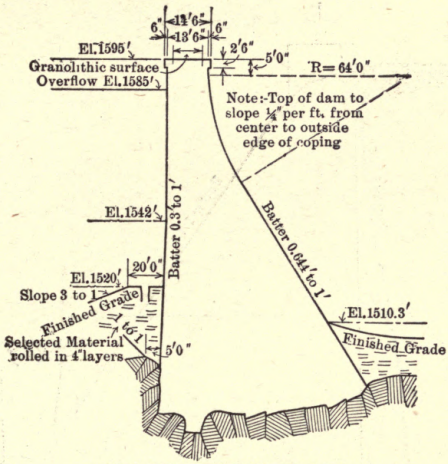
Constr. 1912–1915



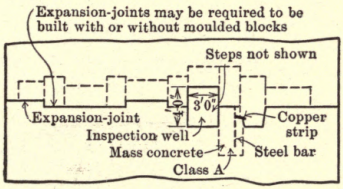
ELEPHANT BUTTE

Rio Grande, New Mexico, U. S A.

Constr. began 1912



TYPICAL SECTION OF DAM



HORIZONTAL SECTION AT EXPANSION-JOINT AT EL.1542

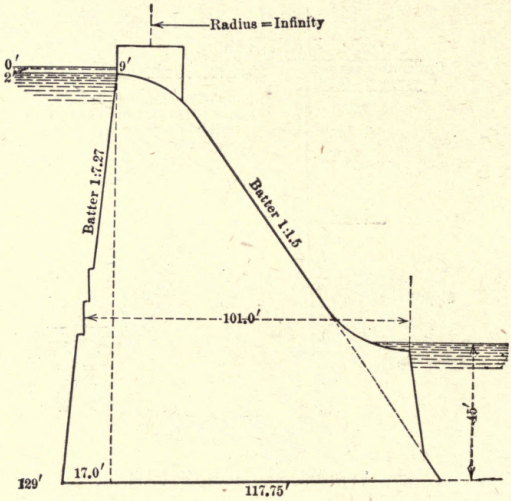
FARNHAM

Mill Brook, Massachusetts, U. S. A.

Completed 1912

Designed for ice-thrust of 10,000 pounds per linear foot of dam length and uplift, intensity due to full head at up-stream edge of joints diminishing uniformly to zero at three-quarters the distance to the down-stream edge of joints.

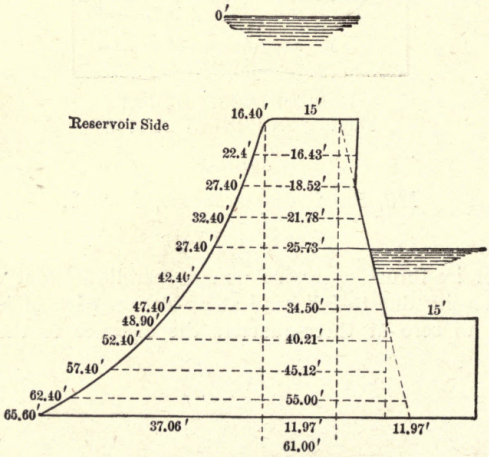
SPILLWAY DAMS



VYRNWY

Vyrnwy River, Wales

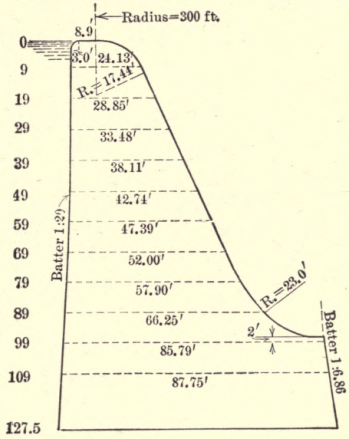
Constr. 1881–1888



BETWA

River Betwa, India

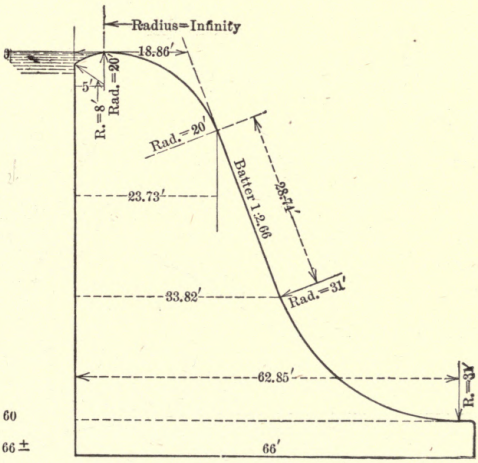
Constr. about 1888



LA GRANGE OR TURLOCK

Tuolumne River, Southern California

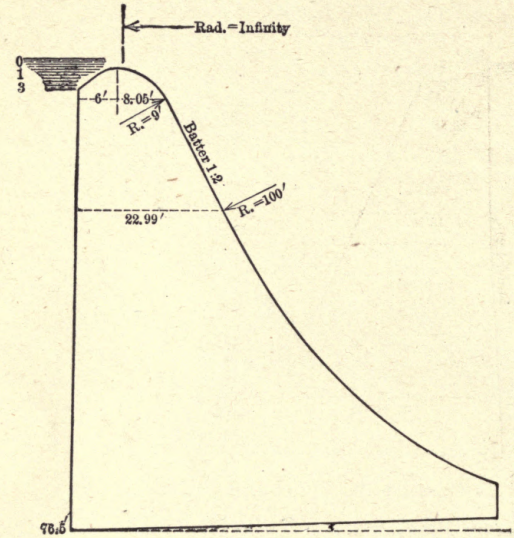
Constr. 1890–1893



COLORADO

Texas, U. S. A.

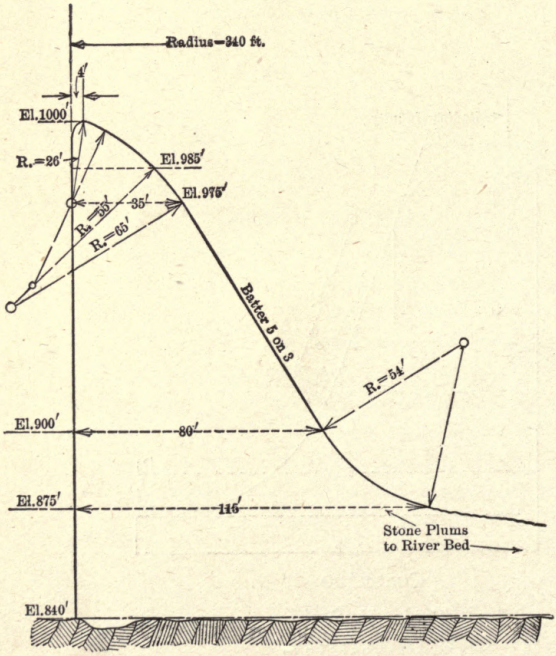
Constr. 1891–1892



SUDBURY

Stoney Brook Branch of Sudbury River, Massachusetts, U. S. A.

Constr. 1894–1897



BIG BEND

Feather River, California, U. S. A.

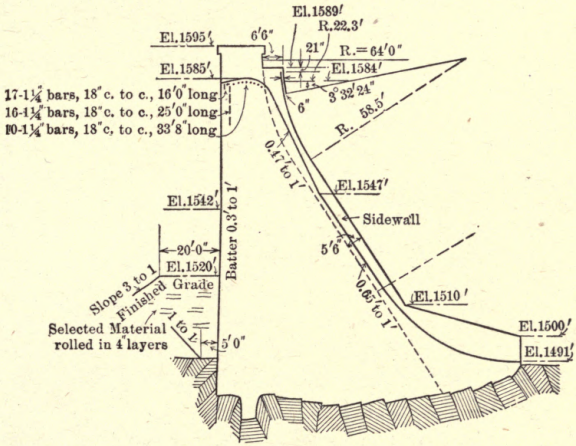
Constr. began in 1909

ASHOKAN WASTE WEIR

New York, U. S. A.

Constr. 1910–1912

(See page 104)



SECTION OF SPILLWAY

FARNHAM

Mill Brook, Massachusetts, U. S. A.

Completed 1912

ASHOKAN DIVIDING WEIR

New York, U. S. A.

Constr. 1912–1914

(See page 103)

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